Development of appropriate structured mathematical models for fish population dynamics along with efficient numerical schemes for the solutions of the developed and other models of engineering problems

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Dedicated to

My Family

&

All my respected Teachers

DECLARATION

I hereby declare that the matter embodied in this thesis is the result of investigation carried out by me in the Department of Mathematics, Shahjalal University of Science and Technology, Sylhet, Bangladesh under the supervision ***Dr. Md. Shajedul karim***, Professor, Department of Mathematics, Shahjalal University of Science and Technology, Sylhet, Bangladesh and co-supervisor ***Dr. Md. Abu Hayat Mithu***, Professor, Department of Industrial and Production Engineering, Shahjalal University of Science and Technology, Sylhet, Bangladesh. This thesis has not been submitted for the Award of any type of Degree, Diploma, Associateship, Fellowship etc. of any other University or Institute.

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ABSTRACT

This thesis is mainly concentrated to obtain numerical solutions of real world problems encountered in continuum mechanics and other branches of sciences like development, management and production oriented agricultural sectors. Appropriate mathematical models are well established for all continuum mechanics problems and hence in such instances a faster algorithm or a technique is the only requirement for obtaining numerical solutions. On the other hand, though the dynamic model approach is a widely applied technique to a number of environmental management and sustainability issues like fisheries management problems still needed to be developed. Therefore, primarily the thesis intends to develop appropriate mathematical models for such real problems and then stresses to obtain their numerical solutions.

More specifically, we intended firstly to develop appropriate mathematical model in order to calculate fish population. As an outcome, we finally present the model by a system of hyperbolic partial differential equations with linear and nonlinear boundary conditions for the calculation of fish population. Secondly, an appropriate model for mathematical estimation of fish production performances is developed for the calculation of fish sizes in different time span depending on initial sizes. Then, as an important integral component, computer codes in FORTRAN that employs Finite Volume Method are developed for obtaining numerical solutions of such models. Afterward, other codes in MATLAB are developed for analyzing and graphical presentation of computed data. Substantiation of the outcomes of the developed models is then established by comparing the computed results with the experimental data.

The versatility and popularity of Finite Element Method (FEM) is well known. The main and important time consuming step in FEM is the formation of all element matrices. Generally, all the elements in global space are transformed into respective contiguous elements in local space by use of isoparametric/

subparametric/ superparametric transformation. For such transformations only all the components of element matrices become integrals of rational functions for the popular quadrilateral as well as for the curved triangular finite elements. The Gaussian quadrature schemes, used in most cases for its simplicity cannot evaluate such rational integrals as it can evaluate the integrals of polynomials of degree/ order (2n-1) with n Gaussian points. For the desired accuracy of the evaluations, more and more Gaussian points are used and eventually that increases the computing time. Therefore, it is an important task to make a proper balance between the accuracy and efficiency of evaluations of numerous rational integrals. So, the thesis concentrates to develop the faster technique by reducing steps of various stages of usual FEM solution procedure for obtaining numerical solutions of numerous boundary value problems governed by hyperbolic, elliptic partial differential equations. For doing so, it stresses to present faster closed form formulae needed to form exactly all types of element matrices for solving such two dimensional boundary value problems encountered in the realm of science and engineering. Since, the faces of finite volume are finite elements so all the formulae are applicable in both FEM, FVM methods. Computer codes compatible with the formulations are also developed accordingly. The efficiency and accuracy of the technique is then demonstrated through application of the formulae in order to obtain the solutions of test problems.

Thus, in brief the Thesis includes: (1) appropriate mathematical model for the calculation of fish population, (2) appropriate mathematical model for the calculation of fish population performances (size of fishes), (3) computer codes employing suitable numerical methods (FVM) for obtaining best approximations of solutions of the developed models, and (4) computer codes based on the developed technique for exact computing all the element matrices efficiently in order to solve numerous two dimensional boundary value problems. All the relevant concepts, mathematical tools, devised; modified; improved algorithms, other related topics and the gradual development of the Thesis work are elaborately described in 7 (seven) chapters.

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Chapter 1

Introduction

Chapter 1

Introduction

This chapter explains the motivation of the work in details. Further, it elaborates necessary and relevant literature review/ survey. Finally, it indicates clearly the objectives and plan of the Thesis.

1.1 MOTIVATION

Fish and fishery products are one of the most widely traded agricultural commodities with exports worth. Fisheries not only meets the demand of the necessary animal protein consumed globally to feed an ever-increasing human population, but also provides an employment for more than two billion people worldwide. This consumption of food fish is increasing as the world population is increasing geometrically, especially in many countries in Asia, Africa and South America. The fisheries are not been able to keep pace with the growing demand, while fisheries in some developed countries are recovering, overfishing is impoverished the state of the ecosystem globally. This threat extensively demands the establishment of sustainable fisheries to explore the challenges of increasing demand of fishmeal for human population.

Bangladesh is a densely populated country, currently with a population of around 160 million people (Worldometer, 2016). This country is an agro-based developing country being endowed with natural fisheries resources. It is fortunate in having an extensive water resource in the form of ponds, natural depressions (i.e., haors and beels), lakes, canals, rivers and estuaries. The consumption of fish food is increasing as the population is increasing geometrically whereas the land and water are decreasing at the same rate.

Therefore, to meet the growing demand, it is necessary to establish sustainable fisheries, or to increase the number of artificial fisheries which is practically impossible due to the consumption of lands by overpopulation.

At present the fisheries sector in Bangladesh plays a significant role for fulfilling the demand of protein, nutrition, employment, poverty alleviation of a large number of unemployed population and foreign exchange earnings. Aquaculture produces about 3.46 million tons of fish, of which about 2 million tons were farmed in the year 2013-14 (Mahmud, 2014). Bangladesh ranks third among the world's largest inland fish producing countries after China and India. Around three quarters of rural households practice some form of freshwater aquaculture covering some 10 million ponds and most of which measure less than 400 m^2 (Ghose, 2014).

Fisheries in Bangladesh are diverse, there are about 795 native species of fish and shrimp in the fresh and marine waters of Bangladesh, and 12 exotic species that have been introduced (Karim, 2003). To meet the present demand and considering future potentials, a large number of fisheries have been established in different parts of the country. Although there seems a huge success in producing large quantity of fish population, the major tasks for fisheries management is to regulate their fishery in such a way as to obtain the maximum benefit from it. However, the fishery is a complex system and it is not easy to interpret the wide range of data that can be obtained about such diverse features as growth rates, harvesting with respect to fishing gear, mortality, immigration and emigration, etc., nor is it easy to predict the effect on the fisheries management. Therefore the owner has to have some options of restricting or encouraging fishing effort, setting catch quotas, and restricting the legal size of the fish captured, and should attempt to control these in such a way as to ensure the fishery meets his management objectives.

Many authors worked on the fish population dynamics based on modelling the growth processes. To devise these models, a considerable amount of work of recording data is required, extending over a prolonged period of time in order to obtain reliable results. Thus, the data collected for the fisheries management to make it possible, in particular, to establish growth and mortality rates. A considerable number of researchers works on the von Bertalanffy's growth equation in order to estimate the both growth and reproduction of fish population. The common practice among researchers who study fish growth is to a priori adopt the von Bertalanffy growth model, which is the most used and ubiquitous equation in the fisheries literature. However, in many cases this model is not supported by the data and many species seem to follow different growth trajectories. The von Bertalanffy growth model assumed asymptotic growth. Most often this model was either strongly supported by the data (with no other substantially supported model) or had very low or no support by the data. The estimation of asymptotic length was greatly model dependent. In this model, it is observed that plotted curve starts from the final size (asymptotic length) and moves towards the initial size of the organism, which resembles a concave downward curve (i.e., exponential decay) in nature. Practically this estimation procedure does not fulfil the requirement because the estimation of fish size based on initial size gives needful idea of harvesting. Moreover, it is must to know the asymptotic length of the organism in order to estimate the size of species in time periods. Consequently, appropriate structured mathematical models for fish population dynamics is one of the prime tasks.

Mathematical models, generally developed by the differential equations require suitable numerical schemes for obtaining approximate solutions. Eventually, the development of suitable numerical schemes is very important as well as essential integral part of developing mathematical models. Several approximate numerical methods have evolved over the years. One of the common methods is the Finite Difference scheme which is an approximation to the governing equation used initially. The solution is formed by writing difference equation for a grid points. In order to improve the accuracy of the solution more grid points are used that is the domain is subdivided by numerous rectangles. By using the Finite Difference

(FD) method many difficult problems may be solved provided the domain can be slipitted by rectangles. Besides that for problems of irregular geometries or unusual specification of boundary conditions, the solution becomes more complex and the method FD generally fails. On the other hand, the Finite Element Method (FEM), Finite Volume Method (FVM), can take care of all these problems, and hence has become more wide spread numerical method for finding solutions of complex structural and non-structural problems encountered in the arena of science and engineering.

Now a day, Finite Element Method (FEM), Finite Volume Method (FVM) are extensively used to solve a large class of engineering problems in stress analysis, heat transfer, electromagnetism and fluid flow etc. Further, FEM has now become a staple for predicting and simulating the physical behavior of science, engineering, medical and business applications. Considering the suitability and utility of the method, numerous research has been already carried out and such research is still going on to improve the efficiency of the method. Thus, the development of efficient numerical methods is the important integral part of developing new as well as existing mathematical models for science and engineering problems.

1.2 LITERATURE REVIEW

This Thesis concentrates first to develop appropriate mathematical models for fish population dynamics and management. Then it employs the suitable numerical methods for obtaining solutions. Further, it introduces all the element matrices needed in Finite Element solution procedure. Finally, it presents a technique for exact and efficient evaluation of element matrices for obtaining solutions of field problems. Consequently, the necessary and relevant literature for the study is further elaborated as in following.

1.2.1 Fish Population Dynamics

Fish population dynamics demonstrates the quantitative changes in the number of individuals in which a given fish population grows and shrinks over time as controlled by some specific factors such as by birth, death, emigration or immigration, fishing gear, sexual differentiation over time etc., and is generally used in the fisheries science in order to determine sustainable yields. As the fishery is a multifaceted system, several mathematical models and techniques are existing for the study of fish population in order to get an idea how much fish be produced by using these parameters.

One of the major tasks for fisheries owners is to attempt to regulate their fishery in such a way as to obtain the maximum benefit from it. Therefore the owner has to have some options of restricting or encouraging fishing effort, setting catch quotas, and restricting the legal size of the fish captured, and should attempt to control these in such a way as to ensure the fishery meets his management objectives. As the fishery is a complex system and it is not easy to interpret the wide range of data that can be obtained about such diverse features as growth rates, behavior with respect to fishing gear, fertility, etc., nor is it easy to predict the effect on the fishery of changes, such as increasing the minimum size at which fish may be taken, or decreasing the fishing effort (Allen, 1975).

A number of authors worked on the fish population dynamics based on modelling the growth processes. Among these researchers, Zhang et al. (2000) proposed and analyzed a model to study the optimal harvesting policy of a stage structured problem and derived necessary and sufficient condition for the coexistence and extinction of species. They considered the stage structure of immature and mature of the first species (their sizes of population were written as x_1; x_2, respectively), and did not consider the stage structure of the second species (its size of population is written as x_3), under some assumptions. In this research they have presented some mathematical relationships among various parameters and pointed out that the product of the relative birth rate and the relative transformation rate was larger than one about the Chinese Alligator.

Song and Chen (2001) deliberated the optimal harvesting policy and stability for a two-species competitive system and derived the conditions for the existence of a globally asymptotically stable positive equilibrium and a threshold of harvesting for the mature population. Dubey et al. (2002) presented a dynamic model for a single-species fishery which depends partially on a logistically growing resource. They showed that both the equilibrium density of fish population as well as the maximum sustainable yield increases as the resource biomass density increases. Later on, Dubey et al. (2003) proposed and analyzed a mathematical model to study the dynamics of a fishery resource system using the Pantryagin's Maximum Principle. They analyzed and propose a mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists of two zones, (a) a free fishing zone, and (b) a reserve zone where fishing is strictly prohibited. They also derived the conditions for the existence of biological and bionomical equilibrium and showed that even if fishery was exploited continuously in the unreserved zone, fish populations can be maintained at an appropriate equilibrium level in the habitat.

Faugeras and Maury (2005a) described a multi-region nonlinear age-size structured fish population model to assess its mathematical posedness based on initial boundary value problem and existence and uniqueness of a positive weak solution. The model was formulated in a generic way so that it can be potentially used for various fish species. They formulated an initial boundary-value problem and proved the existence and the uniqueness of a positive weak solution. They also proved a comparison result which shows that the variations in the mortality rate in each region have consequences on the population of fish in every regions. Moreover, they estimated some badly-known parameters of the model, i.e., growth, mortality and migration rates. Following the same article, later on they (2005b) also developed an advection-diffusion size-structured fish population dynamics model to simulate the skipjack tuna population in the Indian Ocean.

The model was fully spatialized, and movements were parameterized with oceanographical and biological data.

Kar (2006) proposed and analyzed a nonlinear mathematical model to study the dynamics of fishery resource system in an aquatic environment that consists of two zones; a free fishing zone and a reserve zone where fishing is strictly prohibited. He observed that in the absence of predators, even under continuous harvesting, fish population may be maintained at an appropriate equilibrium level. He discussed that the optimal harvesting policy in terms of the total user's cost of harvest per unit of effort equals to the discounted value of the future marginal profit of the effort at the steady-state level. He also noted that if the discounted rate increases, then the economic rent decreases and even may tend to zero if the discounted rate tend to infinity.

Kooten et al. (2010) developed a mathematical model to show the relationship between hatching size and the responses to harvesting mortality using Eurasian perch (*Perca fluviatilis*) as a model organism. The result showed that the hatching was size determined by the dynamics through its effect on the relative strength of cannibalistic mortality and resource competition as mechanisms of population regulation. They categorized two different hatching sizes, e.g. intermediate to large and smaller hatching size in order to establish the model. In populations with intermediate and large hatching size, cannibalistic mortality was an important determinant of population dynamics, and harvesting destabilized the population dynamics. But when hatching size was small, population stability was less sensitive to this type of harvesting. Populations hatching at small size were regulated by competition and harvesting large individuals was affected such populations less. Harvesting was also induced by the growth of very large individuals, but was absent in un-harvested populations. They also showed that the harvesting in cannibalistic lake fish populations could be strongly altered the population dynamics in ways that could only be anticipated on the basis of mechanistic knowledge about how populations were regulated.

Most of these models are setup to preserve the fish species disappearing, to provide assessment of the fish abundance and fishery exploitation in order to determine sustainable yield such that economic purposes or ecological yield (Wentworth et al., 2011). However, most of the models are of weakly coupled hyperbolic partial differential equations with non-local boundary conditions (Aylaj and Noussair, 2010). Dynamic models for the commercial fishing taking into account the economic and ecological factors have been studied extensively.

Link et al. (2011) studied the role of the fishermen's harvesting strategies based on the economic impacts of changes in fish population dynamics. This study shows that the fishing strategy is based on a short optimization period of only two fishing periods, changes in population dynamics have a direct influence on the returns from fishing due to the strong pressure on the stocks applied by the fisheries. If the strategy is based on a longer optimization period, fishing activities may be deferred to allow for stock regrowth, which improves the economic performance of the fisheries. However, in that case the relationship between population dynamics and fishing activities becomes less clear.

Carson et al. (2009) studied the classical Gordon-Schaefer fishery management model by replacing the constant growth rate with a cyclical growth rate. In this study the optimal harvest rate is shown to fluctuate, but the cycle of the harvest rate lags the cycle of the biological growth function with the highest harvest rate occurring after biological conditions start to decline. They also showed that small cyclical fluctuations in one species can result in large fluctuations in the optimal harvest rate of another species if the fish species are interlinked through predator-prey relationships. The dynamic model approach, therefore, is a widely applied suitable technique to a number of environmental management and sustainability issues particularly to the fisheries management problems (Bendor et al., 2009; Martinet et al., 2010; De Lara et al, 2011). Later on, Dubey and Patra (2013) developed a dynamic model for optimal management and utilization of a renewable resource by population. An appropriate Hamiltonian function is formed for the discussion of optimal harvesting of the resources.

To devise these models, a considerable amount of work of recording data is required, extending over a prolonged period of time in order to obtain reliable results. Thus, the data collected for the fisheries management to make it possible, in particular, to establish growth and mortality rates. Mansal et al. (2014) modelled the time evolution of the resources, the fishing effort and the price which is assumed to vary with respect to supply and demand. Solving the variability problems for nonlinear dynamic system relied on the consistency between a controlled dynamic and acceptability constraints applying both to states and decisions of the system.

A considerable number of researchers works has been carried out on the von Bertalanffy's growth equation in order to estimate both the growth and reproduction of fish population (Cloern and Nichols, 1978; Essington et al., 2001; Lester et al., 2004; Taylor et al., 2005; Cailliet et al., 2006).

Katsanevakis and Maravelia (2008) derived the best suitable model for fish growth as a better alternative to a priori using von Bertalanffy growth equation. Among the four candidate models of growth function, namely, von Bertalanffy growth model (VBGM), Gompertz model, Logistic and the Power model; the three former assumed as asymptotic and the latter non-asymptotic growth. In these models, the best model was selected by minimizing the small-sample, bias-corrected form of the Akaike information criterion (AIC). Most often the von Bertalanffy growth model was strongly supported by the data (with no other substantially supported model). The estimation of asymptotic length was greatly model dependent; L_∞ as estimated by VBGM was in every case greater than that estimated by the Gompertz model, which in turn was always greater than that estimated by the Logistic model. For ready understanding and clarity the comparison among different models are shown in Fig.1.1. Therefore, ignoring model selection uncertainty may have serious implications, e.g. when comparing the growth parameters of different fish populations. Multi-model inference by model averaging was recommended to model fish growth, for making robust parameter estimations and dealing with model selection uncertainty.

The underlying principle of the VBGM is that the growth rate of fish tends to decrease linearly with size, as indicated by the equation

$$\frac{dL}{dt} = k_1(L_\infty - L)$$

where k_1 is a relative growth rate parameter (with units year 1) and L_∞ is the asymptotic length (L_∞ has the same biological meaning in Gompertz and Logistic models as well).

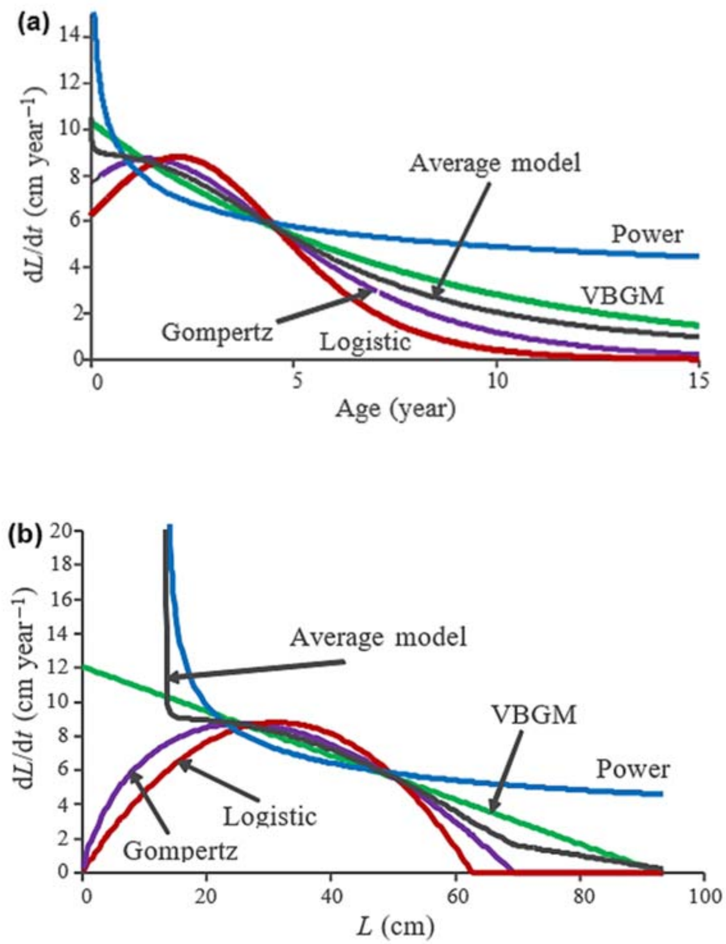


Fig.1.1: Growth rates of the black-bellied angler as estimated by the four candidate models (*e.g.*, von Bertalanffy growth model (VBGM), Gompertz model, Logistic and the Power model) and the average model, in relation to (a) age; and (b) length.

Considering the values of the AICc differences, Akaike weights and asymptotic length, the VBGM was most often selected as the 'best' model used in 34.6% of the cases, followed by the power model (30.1%), the logistic (24.8%) and finally the Gompertz model (10.5%). The frequencies of model applied is summarized in Fig.1.2. The asymptotic length L_∞ as estimated by VBGM was in every case greater than that estimated by the Gompertz model, which in turn was always greater than that estimated by the Logistic model.

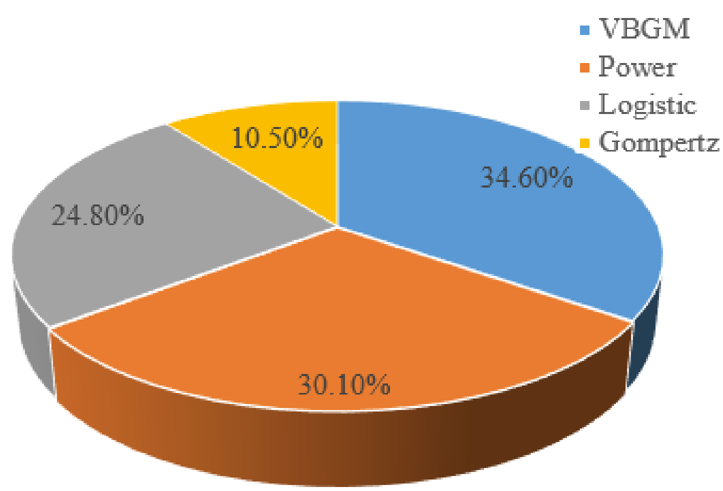


Fig.1.2: The frequencies of model used in the calculation of fish population.

Gómez-Márquez et al, (2008) determined the age and growth of Nile tilapia (*Oreochromis niloticus*) in a tropical shallow lake of Mexico. They used the Taylor's equation which indicated that the age limit or the longevity for the tilapia fish. Hart and Chute (2009) demonstrated a mathematical formulas for estimating von Bertalanffy growth parameters from growth increment data using a linear mixed-effects model that lack explicit age information. Although this approach produces unbiased estimates, it is sometimes difficult to implement and compute the growth and size of fish population in different time spans. Later on, Karna and Panda (2011) introduced multiple length frequency analysis on the total length of the species and proposed the growth curve to estimate the length for different cohorts. They estimated the growth parameter of *D. albida* by using the von Bertalanffy growth equations and using the logistic curves on the proportion of maturation of the females of *D. albida.* Their information were used by most of models of stock assessment to estimate fishing mortality, population of cohorts, population of pawning stock etc.

Eberhardt and Breiwick (2012) modeled the population growth curves for birds and mammals based on different mathematical established formula. They showed that for any practical purposes, the integrated models should be used for species like those considered here (birds and mammals). The Gompertz model may be preferred for some species of fish and for insects. However, the analyses have largely been restricted to data sets that can be fit by the generalized logistic. They showed that the modified logistic,

$$N(t) = [K^{-2} - (K^{-2} - N_0^{-2})e^{-2rt}]^{-1/2}$$

and the ordinary logistic,

$$N(t) = \frac{K}{1 + ce^{-rt}}, \quad c = \frac{K}{N_0} - 1$$

can be fit to a much wider range of data. In a few cases, the exponential,

$$N(t) = N_0 \exp(rt)$$

may appear to give a better fit, but these appear to be largely instances where the data are limited to the early stages of population increase.

In this proposed models, K is the asymptotic value, r is a rate of increase, N_0 represents initial population size, and c is a functions of r and K, and t is the time period in the Gompertz equations.

1.2.2 Numerical Methods (FEM, FVM)

Shape functions are commonly derived in local co-ordinates in local spaces. Generally, the transformation equations are written in terms of shape functions and the original element (in global space) is transformed into its contiguous element in local space. Consequently, all the calculations needed to form the element matrices that is the evaluation of numerous integrals are carried out in local co-ordinate systems (Zienkiewicz and Cheung, 1965; Okabe, 1981a; Okabe, 1981b; Zienkiewicz and Morgan, 1983; Babu and Pinder, 1984; Reddy, 1984; Rathod, 1988; Hacker and Schreyer, 1989; Zienkiewicz and Taylor, 1989; Yagawa and Yashimara, 1990). It is well known that for such transformations (isoparametric, sub-parametric and super parametric) the integrals so encountered to form the stiffness matrix for the general quadrilateral (convex, concave) finite elements are rational integral of bivariate polynomial numerators with bilinear or higher order bivariate expressions denominators (Griffths, 1994; Griffths and Mustoe, 1995; Videla and Carsolaza, 1996; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim., 2001). Evaluation of such integrals defies our analytical skills and therefore we are resort to numerical integration schemes (Zienkiewicz and Cheungy, 1965; Zienkiewicz and Morgan, 1983; Barrett, 1999; Rathod and Karim, 2001). It is astounding to note here that the pleasing advancements in regard to analytical evaluation of such rational integrals for straight sided quadrilateral elements are made by many researchers (Okabe, 1981b; Babu and Pinder, 1984; Rathod, 1988; Yagawa and Yashimara, 1990; Griffths, 1994; Griffths and Mustoe, 1995; Videla and Carsolaza, 1996; Rathod and Islam, 1998; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002). They have identified the drawbacks of numerical integration techniques especially for the Gaussian quadrature schemes. It is also evident from numerous research articles that such analytical integration formulae are applicable for sub-parametric case only and not applicable for higher order isoparametric elements. Besides that these analytical formulae require lot of computational effort and inconvenience for computer coding. Hence, for such difficulties and short comings the numerical integration schemes are still the only instrument for its simplicity and easy incorporation.

Among all the numerical integration schemes, Gaussian quadrature scheme occupies a central role for such evaluations. Complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric/ subparametric/ superparametric elements, the presence of the determinant of the Jacobean matrix in the denominator of the stiffness matrix for which the integrands are rational functions. Many authors (Yagawa and Yashimara, 1990; Lague and Baldur, 1997; Rathod and Islam, 1998; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001) outlined clearly that the usual Gaussian quadrature cannot evaluate exactly such integrals of rational functions as it can evaluate exactly a polynomial of degree $2n$-1 by employing n Gaussian points and weights. Obviously, for the desired accuracy of evaluations the number of Gaussian points and weights are needed to be increased and that increases substantially the computing time. Hence, a proper balance between the accuracy and efficiency is an important task (Yagawa and Yashimara, 1990; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002). Further, an attention is always required to select the order of the integrating rule as it is not yet totally worked out.

Lague and Baldur (1977), Rathod and Karim (2000), Rathod and Karim (2001), have shown that such integrals are also encountered in axisymmetric finite element solution employing triangular elements in a local co-ordinate system. In their (Lague and Baldur) study it is clearly shown that the Gauss quadralure formulae developed by Cowper (1973) are not satisfactory in respect of integration accuracy. Consequently they have developed a technique based on a transformation of the triangular surface into a square for which case the Gauss quadrature formulae have been extensively analyzed and are readily available. It is also demonstrated that for some element geometry the proposed technique is very inefficient to obtain the sufficient degree of accuracy and little assurance is guaranteed if any divergence is evident. These integrals were considered by Mcleod (1978), Andersen and Mcleod (1979), Baart and Mcleod (1983), extending the work of Mcleod (1978), they employed recursive and reduction algorithms to drive analytical, closed-form formulae. They also considered in detail the propagation of errors in the associated recursive formulae. The expressions derived by them ultimately depend upon the evaluation of several integrals, which in turn require a recursive procedure in obtaining the final result. Therefore the evaluation procedure is very lengthy even for one final integral. In their study only one type of integrals of rational integrands had been discussed.

The approach of Rathod and Karim (2000) differs from that of Mcleod (1978), Andersen and Mcleod (1979), Baart and Mcleod (1983), Mcleod and Mitchel (1972) in the sense that they have identified four distinct types of rational integrals depending on the position of intermediate node(s) along the curved side. Accordingly integration formulae they have derived and their application have shown in the calculation of the components of the element stiffness matrices for the second order partial differential equation employing the curved triangular elements and in axisymmetric case with linear triangular elements. A method for exact integration of the polynomials in the convex and concave circular sectors is provided in Silva and Mote (1988). Because of the composition of circular sector on the curved boundary such elements have got limited applications in modelling arbitrary curved domains. Several Authors considered a convex linear quadrilateral element and presented integration formulae. Babu and Pinder (1984), Griffths (1994), Griffths and Mustoe (1995), Hacker and Schreyer (1989), Okabe (1981), Rathod (1988), Videla (1996) have successfully attempted to drive the explicit finite elements relations for general convex linear quadrilateral finite elements. A good overview of numerical integration schemes which employ Gaussian quadrature is given by Yagawa and Yashimara (1990). They have developed a numerical integration scheme based on symbolic manipulation, which can dramatically reduce the computing time of the usual numerical integration and give adequate integration accuracy by increasing integration points. They have shown that the evaluation of the coefficients of the bivariate polynomial in numerator with linear denominator requires relatively large amount of time in the total estimation of the stiffness matrix. The computation of these coefficient has to be improved in order to reduce the total computational time. Karim (2001), Rathod and Karim (2002) have presented methods for computing the components of element stiffness matrix in an efficient way employing the exact integration formulae encountered for planar quadrilateral and curved triangular elements. Karim (2001) mainly contributed with special attention on the four-node, six-node and ten-node triangular elements with two straight sides and one curved side.

1.3 OBJECTIVES

From the aforementioned literatures it is found that the birth (or recruitment) and mortality rate, harvesting, time, emigration or immigration rate, setting catch quotas, restricting the legal size of the fish capture, etc. are the controlled parameters of fish population growth. Therefore, the primary objective of this thesis is to develop:

- A mathematical model with its validation, stability and convergence which is useful for the different fish species.

Indeed, most fish population share specific characteristics, which need to be taken into account in order to model their dynamics in a realistic manner. In this model the population dynamics in which both size and time would be taken as structure variables to account for growth, mortality, movements of fish, environmental variability and variable distribution of fishing effort might be considered as input variables within the multi-regions.

The second and most important goal is to:

- Employ the suitable numerical methods for obtaining numerical solution i.e. population size periodically.

After that, with the intention to estimate the final fish size (asymptotic length), the von Bertalanffy's growth equation needs to modify and utilize based on the developed equation for the fish species. Therefore, the objective of the work will be extended to:

- Obtain the growth equation modifying the von Bertalanffy's equation for the calculation of population size based on initial size.

After modifying the von Bertalanffy's growth equation, the modified equation will be tested. The number of total fish population and the fish size at different time spans will be calculated. Then, the fourth objective will be:

- Calculated results from the developed models will be compared to the experimental results of population production for the complete validation.

As the important integral part of the work, the efficient technique to evaluate exactly the element matrices is to be developed for obtaining solutions of structured, non-structured field problems.

1.4 ORGANIZATION OF THE THESIS

To achieve the objectives of this Thesis explain in section 1.3, the plan of the work is as follows. The gradual development of the Thesis work spreads over 7 (seven) chapters:

Chapter 1 clearly explores the motivation of the work in details. Further, it elaborates necessary and relevant literature review/ survey and indicates clearly the objectives of this Thesis.

Chapter 2 includes basic concepts, mathematical preliminaries, and historical back ground of numerical methods (FEM, FVM) and existing mathematical models of continuum mechanics problems. Indications of necessary development of formulae are also given and importance is focused.

Chapter 3 is concerned with the development of a mathematical model for better management of fisheries resources. This model is nonlinear because all the parameters are chosen size dependent and age independent. The existence and uniqueness of the solution is proved. By using the finite volume method the continuous problem is discretized and then upwind explicit scheme is developed.

Consequently, this model approaches the problem by the upwind explicit scheme for which the consistence and stability are established. A computer code in FORTRAN® compatible with the formulations is developed for obtaining numerical solutions and results are shown graphically.

Chapter 4 is devoted to present and analyze a generic mathematical formula of a single-region size structured model which is useful for the fish production estimation. The well-known and widely used von Bertalanffy's growth equation for estimation of fish size is modified and utilized here with the initial size of the fish species. The model so developed for fish population calculation utilizes initial size, birth, growth, mortality rates and the arbitrary constant of modified von Bertalanffy's growth equation as input variables. Then, the number of total fish population and the fish size at different time spans are computed. The accuracy of the mathematical model is substantiated through the comparison of the computed and the experimental results.

Chapter 5 mainly intended to present general forms of all element matrices encountered in Finite Element solution procedure of science and engineering problems. For such formations it reviews the finite element equations of two engineering problems in details and focuses the necessity of exact computations of all the components of element matrices.

Chapter 6 concentrated to present close form formulae for computing all type of element matrices in an efficient way. Primarily, efforts endured to derive shape functions in global coordinates in easiest way. Then, a technique, for the first time is developed that utilizes a matrix G formed by the nodal coordinates of the element and its inverse matrix H together with integral values of monomials over the element for all sort of derivations and computations. It is clearly shown that the technique does not require any transformations from global to local spaces and consequently the transformation of the integrals along with their numerical evaluations are not at all required. Hence, the employment of the method reduces many steps of traditional approach of Finite Element solution procedure and hence the computing time as well as effort reduces substantially. On the other hand the exact integration formula to evaluate the integrals of monomials over the element is incorporated with the method for computations of element matrices. Thus, exact computation of element matrices with less computational effort and as a result proper balance between accuracy and efficiency is ensured. A suitable computer code in MATLAB® compatible with formulation is developed and then demonstrated thoroughly by showing its application to solve many application examples of continuum mechanics problems.

Finally, **chapter 7** focuses on the main conclusions of the research work. All the developed computer codes are appended.

Chapter 2

Some Preliminaries

Chapter 2

Some Preliminaries

This chapter mainly intended to include (i) some basic definitions, well known results and theorems concisely from a range of mathematical origins-particularly from analysis, (ii) brief description of population dynamics of fisheries and relevant definitions, (iii) concept of initial value problems, boundary value problems and eigen value problems, (iv) existing models of field problems and useful formulae, (v) brief history, merits and demerits of numerical methods and, (vi) derivation of element equations of field problems for clarity and ready reference.

2.1 BASIC DEFINITIONS AND THEOREMS

2.1.1 Functional spaces

Definition 2.1: (Linear (vector) spaces)

A *linear space* over a field K is a V equipped with maps $\oplus : V \times V$ and $\mathbb{R}: K \times V \to V$ with the properties

(1) $x \oplus y = y \oplus x$ for all $x, y \in V$ (addition is commutative);

(2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in V$ (addition is associative);

(3) There is an element $0 \in V$ such that $0 \oplus x = x \oplus 0 = 0$ for all $x \in V$ (a zero element);

(4) For each $x \in V$ there is a unique element $-x \in V$ with $x \oplus (-x) = 0$ (additive inverses) (notice that $(V.+)$ therefore forms an abelian group)

(5) $\|\alpha x\| = \alpha \|x\|$ for all $x \in V$ and $\alpha \in K$.

(6) $(\alpha + \beta).x = \alpha.x + \beta.x$ for all $\alpha, \beta \in K$ and $x \in V$ (scalar multiplication distributes over scalar addition);

(7) $\alpha.(x \oplus y) = \alpha.x \oplus \beta.y$ for all $\alpha \in K$ and $x, y \in V$ (scalar multiplication distributes over vector addition);

(8) $1.x = x$ $\forall\, x \in V$ where 1 is the multiplicative identity in the field *K*.

Definition 2.2: (Norms)

A *norm* on a vector space is a way of measuring distance between vectors. A norm on a linear space *V* over *K* is a non-negative function $\|\mathbb{R}\| : V \to \mathbb{R}$ with the properties that

(1) $\|x\| = 0$ if and only if $x = 0$ (positive definite);

(2) $\|x + y\| \le \|x\| + \|y\| 0$ for all $x, y \in V$ (triangle inequality);

(3) $\|\alpha x\| = \alpha \|x\|$ for all $x \in V$ and $\alpha \in K$.

Definition 2.3:

A linear space *V* together with a norm is called a *normed linear space* or simply a *normed space*.

Theorem 2.1:

Let $f : G \to \mathbb{R}$ be a continuous function defined on a set *G* containing a neighborhood $\{(x, y) \| \|(x, y) - (x_0, y_0)\|_\infty < e\}$ of (x_0, y_0) for some $e > 0$. Suppose that *f* satisfies a *Lipschitz condition* of the form

$$|f(x, y) - f(x, \tilde{y})| \le M |y - \tilde{y}|$$

In the variable *y* on *G*. Then there is an interval on which the ordinary differential equation

$$\frac{dx}{dy} = f(x, y)$$

has a unique solution $y = \phi(x)$ satisfying the initial condition $\phi(x_0) = y_0$.

Definition 2.4:

$C^l(\Omega)$ is the class of functions in Ω such that $u(x)$ and $\partial^\alpha u, |\alpha| \le l,$ are continuous in Ω.

Definition 2.5:

$C_0^\infty(\Omega)$ is the class of functions $u(x)$ in Ω such that

(a) $u(x)$ is infinitely smooth, which means that $\partial^\alpha u$ is uniformly continuous in $\bar{\Omega}, \forall\alpha;$

(b) $u(x)$ is compactly supported: *sup u* is compact subset of Ω.

Definition 2.6: (Banach space)

$L_q(\Omega), 1 \le q \le \infty,$ is the set of all measurable function $u(x)$ in Ω such that the norm

$$\|u\|_{q,\Omega} = \left(\int_\Omega |u|^q \, dx\right)^{1/q}$$

is finite. $L_q(\Omega)$ is a *Banach space*.

Definition 2.7:

A complete normed linear space is called a *Banach space.*

Definition 2.8:

$L_\infty(\Omega)$ is the set of all bounded measurable function in Ω; the norm is defined by

$$\|u\|_{\infty,\Omega} = ess \sup_{x\in\Omega} |u(x)|$$

Definition 2.9:

$L_\infty(\Omega)$ is the *Banach space* of all functions in $\bar{\Omega}$ such that $u(x)$ and $\partial^\alpha u$ with $|\alpha| \le l$ are *uniformly continuous* in $\bar{\Omega}$ and the norm

$$\|u\|_{C^l(\bar{\Omega})} = \sum_{|\alpha|\le l} \sup_{x\in\Omega} \left|\partial^\alpha u(x)\right|$$

is finite. If $l = 0$, we denote $C^0(\bar{\Omega}) = C(\bar{\Omega})$.

Definition 2.10: (Hilbert spaces)

A complex linear space H is called a *Hilbert space* if there is a complex valued function $(\cdot,\cdot) : H \times H \to \mathbb{R}$ with the properties

(i) $(x, x) \ge 0$ and $(x, x) = 0$ if and only if $x = 0$;

(ii) $(x + y, z) = (x, z) + (y, z)$ for all $x, y, z \in H$;

(iii) $(\lambda x, y) = \lambda(x, y)$ for all $x, y \in H$ and $\lambda \in \mathbb{R}$;

(iv) $(x, y) = (y, x) + (y, z)$ for all $x, y \in \mathbb{R}$;

If only properties (i), (ii), (iii), (iv) hold then $(H(\cdot,\cdot))$ is called an *inner product space.*

The function $(\cdot,\cdot)$ is called *inner* or *scalar* product, and so a Hilbert space is a complete *inner product space*.

If the scalar product is real valued on a real linear space, then the properties determine a *real Hilbert space*; all the results below apply to these properties.
A *Hilbert space* is a scalar product for which the corresponding normed space is complete.

Definition 2.11:

Let H be a Hilbert space, $L^2(\mathfrak{D}, H)$ denotes the space of measurable functions of $\mathfrak{D}$ having values in H such that

$$\|u\|_{L^2(\mathfrak{D},H)} = \left(\int_{\mathfrak{D}} |u|_H^2 \, dx\right)^{1/2} < \infty \qquad (2.1)$$

Definition 2.12:

We know that the spaces $W_2^l(\Omega), (l \in \mathbb{R})$ are Hilbert space: $W_2^l(\Omega) = H^l(\Omega)$. Let $\Omega = \mathbb{R}^n$, we can use the Fourier transform and express the norm in $W_2^l(\Omega) = H^l(\Omega)$ in terms of the Fourier image. Let $u \in H^l(\mathbb{R}^n)$, Consider the Fourier image

$$\hat{u}(\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} u(x) e^{-ix\xi} dx \qquad (2.2)$$

Then

$$u(\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} \hat{u}(x) e^{ix\xi} dx \qquad (2.3)$$

Theorem 2.2:

Let $u \in H^l(\mathbb{R}^n)$, $v \in H^{-l}(\mathbb{R}^n)$ (H^{-l} Duality of H^l) and let $u_j, v_j \in C_0^\infty(\mathbb{R}^n)$, $u_j \xrightarrow{j\to\infty} u$ in $H^l(\mathbb{R}^n)$, $v_j \xrightarrow{j\to\infty} v$ in $H^{-l}(\mathbb{R}^n)$. Then there exists the limit

$$\lim_{x\to\infty} \int_{\mathbb{R}^n} u_j(x) \overline{v_j(x)} dx.$$

We denote this limit by $\int_{\mathbb{R}^n} u_j(x) \overline{v_j(x)} dx.$ We have

$$\left| \int_{\mathbb{R}^n} u_j \overline{v_j} dx \right| \le \|u\|_{H^l} \, \|v\|_{H^{-l}} \qquad (2.4)$$

Theorem 2.3:

Let $l(u)$ be a linear continuous functional on $H^l(\mathbb{R}^n)$. Then there exists unique element $v \in H^{-l}(\mathbb{R}^n)$, such that

$$l(u) = \int_{\mathbb{R}^n} u_j \overline{v_j} dx, \quad \forall u \in H^l(\mathbb{R}^n) \qquad (2.5)$$

and

$$\|l\| = \|v\|_{H^{-l}} \qquad (2.6)$$

Definition 2.13: (Sobolev space)

Sobolev spaces consist of functions whose weak derivatives belong to L^p . These spaces provide one of the most useful settings for the analysis of PDEs.

Definition 2.14:

Suppose that Ω is an open set in $\mathbb{R}^n, k \in \mathbb{R}$, and $1 \le p \le \infty$. The *Sobolev space* $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that $\partial^\alpha f \in L^p(\Omega)$ for $0 \le |\alpha| \le k$. We write $W^{k,p}(\Omega) = H^k(\Omega)$.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha f|^p \, dx \right)^{1/p} \tag{2.7}$$

for $1 \le p < \infty$ and

$$\|f\|_{W^{k,p}} = \max_{|\alpha| \le k} \sup_\Omega |\partial^\alpha f| \tag{2.8}$$

As usual, we identify functions that are equal almost everywhere. We will use these norms as the standard ones on $W^{k,p}(\Omega)$, but there are other equivalent norms *e.g.*,

$$\|f\|_{W^{k,p}} = \sum_{|\alpha| \le k} \left(\int_\Omega |\partial^\alpha f|^p \, dx \right)^{1/p} \tag{2.9}$$

$$\|f\|_{W^{k,p}} = \max_{|\alpha| \le k} \left(\int_\Omega |\partial^\alpha f|^p \, dx \right)^{1/p} \tag{2.10}$$

The space $H^k(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{|\alpha| \le k} \int_\Omega (\partial^\alpha f)(\partial^\alpha g) dx \tag{2.11}$$

We will consider the following properties of Sobolev spaces in the simplest settings.

(1) Approximation of Sobolev functions by smooth functions;

(2) Embedding theorems;

(3) Boundary values of Sobolev functions and trace theorems;

(4) Compactness results.

Theorem 2.4: (Banach Fixed point Theorem)

Let E be the Banach space and F closed subset of E. Let $\mathcal{F}$ be a contraction mapping for F into F. Then there exists a unique $\mathbf{p} \in \mathcal{F}$ such that $\mathcal{F}(\mathbf{p}) = \mathbf{p}$.

Theorem 2.5: (Riesz Theorem)

Let H be Hilbert space and $l(u)$, $u \in H$, be a continuous linear function on H. Then there exists such element $v \in H$ that $l(u) = (u,v)_H$. This element v is unique and $\|l\| = \|v\|_H$.

Theorem 2.6: (The Lax-Milgram theorem)

Let V be a Hilbert space with norm $(.,.)_V$ and scalar product $\|.\|_V$ and assume that A is a bilinear functional and L is linear functional that satisfy:

i. A is symmetric, i.e. $A(v,w) = A(w,v) \quad \forall v,w \in V$;

ii. A is V-elliptic, i.e. $\exists\, \alpha > 0$ sust that $A(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$;

iii. A is continuous, i.e. $\exists\, C \in \mathbb{R}$ sust that $|A(v,w)| \leq C \|v\|_V \|w\|_V$; and

iv. V is continuous, i.e. $\exists\, \Lambda \in \mathbb{R}$ sust that $|L(v)| \leq \Lambda \|v\|_V \quad \forall v \in V$.

Then there is a unique function $u \in V$ such that $A(u,v) = L(v) \quad \forall v \in V$, and the stability estimate $\|u\|_V \leq \Lambda/\alpha$ holds.

Theorem 2.7: (The Lax equivalence theorem)

If the Numerical scheme is well posed and the approximation scheme is consistence then stability is equivalent to convergence.

2.2 POPULATION DYNAMICS OF FISHERIES

A fishery is an area with an associated fish or aquatic population which is harvested for its commercial or recreational value. Fisheries can be wild or farmed. Population dynamics describes the ways in which a given population grows and shrinks over time, as controlled by birth, death, and migration. It is the basis for understanding changing fishery patterns and issues such as habitat destruction, predation and optimal harvesting rates. The **population dynamics of fisheries** is used by fisheries scientists to determine sustainable yields.

The basic accounting relation for population dynamics is the BIDE (Birth, Immigration, Death, and Emigration) model, shown as:

$$N_1 = N_0 + B - D + I - E$$

$$N_{t+\Delta t} = N_t + B - D + I - E$$

where N_t is the population size at time t, $N_{t+\Delta t}$ is the population size some time interval Δt, and B is the number of individuals born, D the number that died, I the number that immigrated, and E the number that emigrated in the time period.. While immigration and emigration can be present in wild fisheries, they are usually not measured.

A fishery population is affected by three dynamic rate functions:

- **Birth rate or recruitment:** Recruitment means reaching a certain size or reproductive stage. With fisheries, recruitment usually refers to the age of a fish can be caught and counted in nets.
- **Growth rate:** This measures the growth of individuals in size and length. This is important in fisheries where the population is often measured in terms of biomass.
- **Mortality:** This includes harvest mortality and natural mortality. Natural mortality includes non-human predation, disease and old age.

If these rates are measured over different time intervals, the **harvestable surplus** of a fishery can be determined. The harvestable surplus is the number of individuals that can be harvested from the population without affecting long term stability (average population size). The harvest within the harvestable surplus is called **compensatory mortality**, where the harvest deaths are substituting for the deaths that would otherwise occur naturally. Harvest beyond that is **additive mortality**, harvest in addition to all the animals that would have died naturally.

Definition 2.15: (Growth rates)

Any model of a size-structured population requires modelling of the growth rate of individuals through time. This problem has been tackled in great detail by fisheries biologists, largely because ‘growth overfishing’ (fishing too hard so that fish are caught too soon when they are still too small) is one of the major practical considerations in fisheries management (Hilborn & Walters, 1992). The first problem in parameterizing a growth model is to decide on a suitable functional form. There is a plethora of possible functional forms in the fisheries literature (Ricker, 1979; Kaufmann, 1981; Schnute, 1981). The most commonly used growth model, by a considerable margin, is the *von Bertalanffy model* (von Bertalanffy, 1938),

$$L_t = L_\infty[1 - e^{-K(t-t_0)}]$$

Here, L_t is the length of an individual at time t, L_∞ is the asymptotic length, K is the age-specific growth rate, and t_0 is a shift parameter to allow the extrapolated size at age 0 to be nonzero. An additional allometric parameter b is also frequently added, particularly if weight w, rather than length, is modelled:

$$w_t = w_\infty[1 - e^{-K(t-t_0)}]^b$$

Two quite different sorts of data may be used to estimate a growth function for a population (Francis, 1988). Age–size data consist of records of animal size, together with their age. Recapture data are obtained by measuring, marking, releasing and then recapturing animals, so that the growth over the period at liberty can be measured.

Age–size data can obviously only be used if a means of ageing animals is available. The technical difficulties in ageing wild animals are substantial, but are beyond the scope of this study. Maximum likelihood estimation for any of the models is relatively straightforward, provided that the ages are accurate, and that the errors can be assumed to be additive (Kimura, 1980). Any nonlinear least squares regression routine will produce maximum likelihood estimates of the parameters in a nonlinear model, provided the errors are additive, independent and normally distributed with a constant variance. A problem that is likely to occur in growth data is that larger errors may be associated with larger sizes, and thus older animals. A transformation to stabilize the error variance should be applied, if necessary, before using the nonlinear regression procedure. The parameters estimated by this approach are the means of the parameters for all members of the sampled population. In particular, if an asymptotic size is estimated, it is the mean size of 'old' individuals in the population, and some animals would be expected to reach a larger size (McCallum, 2000).

Definition 2.16: (Torque)

Consider a particle of mass m moving about an axis in a circular path of radius r. Let an external force F act on the particle along the tangent to the circular path.
The momentum of the force = Fr
This momentum of force is also called torque represented by the symbol τ.

$$\tau = Fr \quad [\because\ a = \alpha\, r]$$

But,
$$F = ma$$
$$= m\alpha r$$
$$\therefore\ \tau = m\alpha r^2$$

But,
$$I = mr^2$$
$$\therefore\ \tau = I\alpha$$

Hence torque is equal to the product of moment of inertia and angular acceleration.

Torque is also defined as the rate of change of angular momentum.

$$\tau = \frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha$$

It is vector quantity. Its dimension are [ML^2T^{-2}].

Its units are kg-m^2/s^2 or N/s^2

2.3 TORSION CONSTANT

In 1820, the French engineer A. Duleau derived analytically that the torsion constant of a beam is identical to the second moment of area normal to the section , which has an exact analytic equation, by assuming that a plane section before twisting remains planar after twisting, and a diameter remains a straight line. Unfortunately, that assumption is correct only in beams with circular cross-sections, and is incorrect for any other shape where warping takes place.

For non-circular cross-sections, there are no exact analytical equations for finding the torsion constant. However, approximate solutions have been found for many shapes. Non-circular cross-sections always have warping deformations that require numerical methods to allow for the exact calculation of the torsion constant.

The torsional stiffness of beams with non-circular cross sections is significantly increased if the warping of the end sections is restrained by, for example, stiff end blocks.

For a beam of uniform cross-section along its length:

$$\theta = \frac{TL}{JG}$$

where, θ is the angle of twist in radians, T is the applied torque, L is the beam length, J is the torsional constant, G is the Modulus of rigidity (shear modulus) of the material

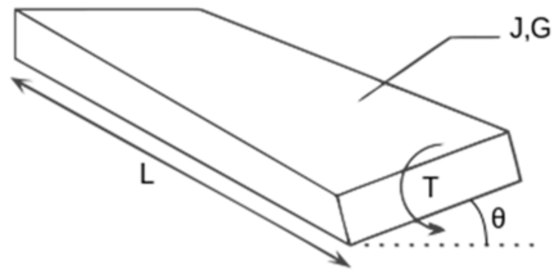


Fig.2.1: A uniform beam

Examples for specific uniform cross-sectional shapes

Solid Circular section (Fig.2.1*a*)**:**

$$J = \frac{1}{4}\pi r^4$$

where r is the radius.

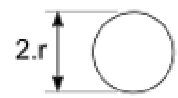


Fig.2.1*a*

Solid elliptical section (Fig.2.1*b*)**:**

$$J \approx \frac{\pi a^3 b^3}{a^2 + b^2}$$

where a is the major radius and b is the minor radius.

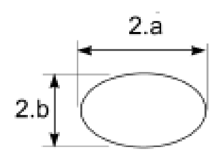


Fig.2.1*b*

Solid square section (Fig.2.1*c*)**:**

$$J \approx 2.25a^4$$

where a is half the side length.

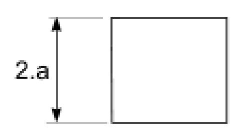


Fig.2.1*c*

Solid rectangular section (Fig.2.1*d*)**:**

$$J \approx ab^3\left(\frac{1}{3}-0.21\frac{b}{a}\left(1-\frac{b^4}{12a^4}\right)\right)$$

where a and b are the side of the rectangle.

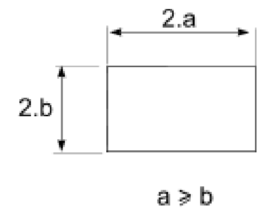


Fig.2.1*d*

Solid Equilateral Triangle (Fig.2.1*e*)**:**

$$J=\frac{a^4\sqrt{3}}{80}$$

where a is the side of the triangle.

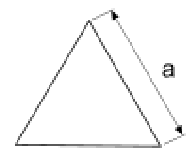


Fig.2.1*e*

Isoscleles triangle (Fig.2.1*f*)**:**

For $39^\circ < \alpha < 82^\circ$, $J=\dfrac{a^3b^3}{15a^2+20b^2}$

and

For $82^\circ < \alpha < 120^\circ$, $J=0.0915b^4\left(\dfrac{a}{b}-0.8592\right)$

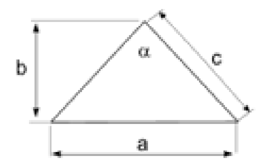


Fig.2.1*f*

These are approximate formulas are only accurate at $\alpha = 60^\circ$ and $\alpha = 90^\circ$.

Solid Hexagon section (Fig.2.1*g*)**:**

$$J=0.1154s^4$$
$$=0.0649d^4$$

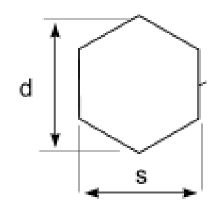


Fig.2.1*g*

Thin tube of uniform thickness (Fig.2.1*h*)**:**

$$J=\frac{4A^4t}{U}$$

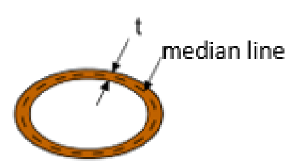


Fig.2.1*h*

where U is length of the median line and A is approximate area enclosed by median line.

Hollow elliptical section (Fig.2.1*i*):

$$J = \frac{\pi a^3 b^3 t}{a^2 + b^2}(1 - q^4), \text{ where } q = \frac{a_i}{a} = \frac{b_i}{b}$$

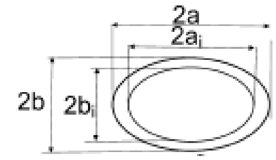
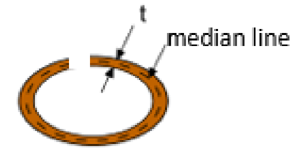


Fig.2.1*i*

Thin open tube of uniform thickness (Fig.2.1*j*):



$$J = \frac{Ut^2}{3}$$

where U is length of the median line.

Fig.2.1*j*

Doubly-symmetric Wide-Flange shapes (W-shapes and I-Beams) (Fig.2.1*k*):

$$J = \frac{2bt^3 + d'w^3}{3}, \text{ where } d' = d - t$$

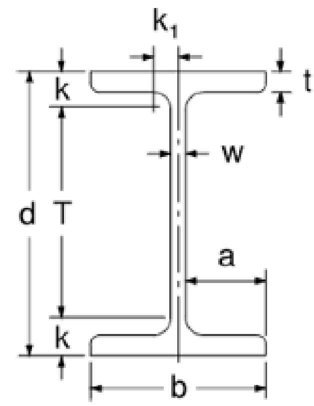


Fig.2.1*k*

Channels (Fig.2.1*l*):

$$J = \frac{2b't^3 + d'w^3}{3}$$

where $d' = d - t, b' = b - \frac{w}{2}$

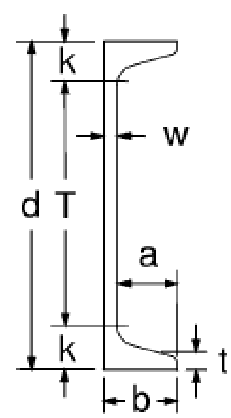


Fig.2.1*l*

Angles (Fig.2.1*m***):**

$$J = \frac{(b' + d')t^3}{3}$$

where $d' = d - \frac{t}{2}, b' = b - \frac{t}{2}$

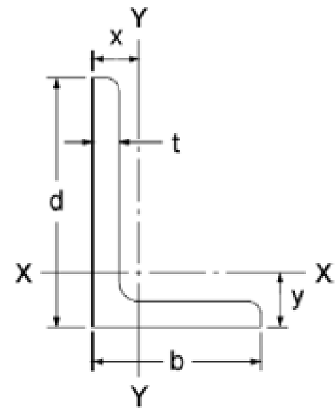


Fig.2.1*m*

Mono symmetric Wide-Flange Shapes (Fig.2.1*n***):**

$$J = \frac{b_1 t_1^3 + b_2 t_2^3 + d'w}{3}$$

where $d' = d - \frac{t_1 + t_2}{2}$

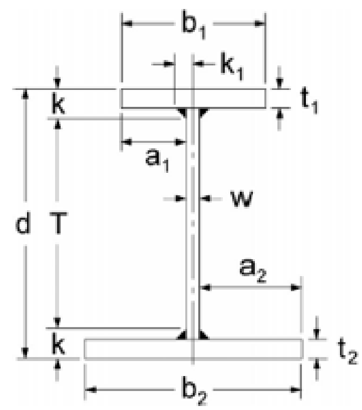


Fig.2.1*n*

Hollow Structural Sections (HSS), Round (Fig.2.1*o***):**
Saint Venant torsion constant (valid for any thickness):

$$J = 2l = \frac{\pi}{32}[d^4 - (d - 2t)^4]$$

where *d* is the outer diameter, *l* is the moment of inertia and *t* is the wall thickness.

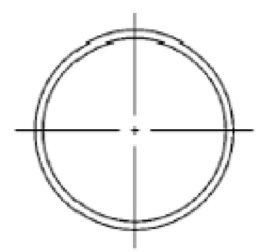


Fig.2.1*o*

Hollow Structural Sections (HSS), Square and Rectangle Saint Venant torsion constant (valid for thin-walled sections, $b / t \geq 10$) **(**Fig.2.1*p***):**

$$J \approx \frac{4A_P^2 t}{p}$$

Mid-contour length:

$$p = 2[(d-t)+(b-t)] - 2R_c(4-\pi)$$

Enclosed area:

$$A_p = (d-t)(b-t) - 2R_c^2(4-\pi)$$

Main corner radius:

$$R_c = \frac{R_0 + R_1}{2} \approx 1.5t$$

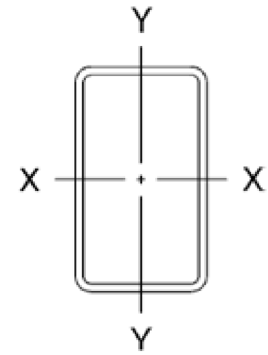


Fig.2.1*p*

where *d* and *b* are longer and shorter outside dimensions, respectively and *t* is the wall thickness, R_0 and R_1 are the outer and inner corner radii taken equal to 2*t* and *t*, respectively.

2.4 SHAPE FUNCTIONS (INTERPOLATION FUNCTIONS)

In finite element analysis using the displacement model, one assumes the variation of displacement within an element since the 'true' variation of displacement is not known. In general, in higher mathematics, it is necessary in many situations to deal with functions whose analytical form is either totally unknown or else is of such a nature that the function cannot easily be subjected to such operations as may be required. In either case, it is desirable to replace the given function by another function, which can be more easily handled. This operation of replacing or representing a given function by simpler one is known as interpolation in a broad sense. In finite element literature it is also referred to as "**Shape Functions**".

There are two types of interpolation functions: (1) Lagrange interpolation and (2) Hermite interpolation. In the Lagrange interpolation, which is widely used in practice, the assumed function takes on the same values as the given function at specified points. In the Hermite type of function, the slopes of the function also take the same value as the given function at specified points. Lagrange's interpolation formula can derive the Lagrange type shape function more directly. An easy and systematic method of generating shape functions of any order can be achieved by simple products of appropriate polynomials in two or three co-ordinates. Now a days lower and higher order shape functions for all finite elements are available. In a recent study, Rathod and Sridevi (2000) provided a solid foundation for the derivations of shape functions and presented them explicitly for complete Lagrange element family in two - dimensions.

For any specified element the values of shape function N_i (say) gives a unit value at the node i unity and zero value at the other nodes of the element. The variation of field variable u (say) in the element can be described as the sum of all shape functions each multiplied by the Corresponding nodal displacement i.e. $u = \sum_{i=1}^{n} u_i N_i$. Hence, shape function plays the most important role in the finite element solution procedure.

2.5 ISOPARAMETRIC ELEMENTS

For the analysis of physical problems of complex shapes involving curved boundaries, simple triangular or rectangular elements are no longer sufficient. This has led to the development of elements of more arbitrary shapes are called **isoparametric** elements. The concept of **isoparametric** elements is based on the transformation of the parent element into local or natural co-ordinate system to an arbitrary shape in the global co-ordinate system. A convenient way of expressing the transformation is to make use of the shape functions of the rectilinear elements in their natural co-ordinate system and the nodal values of the co-ordinates. Thus the global *x*, *y* co-ordinates of a point in an element may be expressed as

$$x = \sum_{i=1}^{n} x_i N_i^{'} \quad , \qquad\qquad y = \sum_{i=1}^{n} y_i N_i^{'}$$

where $N_i^{'}$ are the shape functions of the parent rectilinear element and x_i ,y_i are nodal co-ordinates of the element. Thus, the shape functions $N_i^{'}$ used in the above transformation help us to define the geometry of the element in the global co-ordinate system. If these shape functions $N_i^{'}$ are same as the shape functions N_i

used to represent the variation of displacement i.e. $u = \sum_{i=1}^{n} u_i N_i$ in the element, these elements are called **Isoparametric** elements. In addition, in cases where the geometry of the elements is defined by shape functions of order higher than that for representing the variation of displacements, the elements are called '**Superparametric**'. Similarly if more nodes are used to define displacement compared to the nodes used to represent the geometry of the elements, then they referred to as '**Subparametric**' elements. Such elements generally found to be more often of use in practice.

2.6 CHARACTERISTICS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

For a first-order partial differential equation (PDE), the method of characteristics discovers curves (called characteristic curves or just characteristics) along which the PDE becomes an ordinary differential equation (ODE). Once the ODE is found, it can be solved along the characteristic curves and transformed into a solution for the original PDE.

Definition 2.17:

A one-step finite difference scheme approximating a partial differential equation is a *convergent* scheme if for any solution to the partial differential equation, $u(t,x)$ and solution to the finite difference scheme v_i^n, such that v_i^0 to $u_0(x)$ as $i\Delta x$ converges to x, then v_i^n converges to $u(t,x)$ as $(n\Delta t, i\Delta x)$ converges to (t, x) as $\Delta t, \Delta x$ converge to 0.

Definition 2.18:

Given a partial differential equation $Pu = f$ and a finite difference scheme, $P_{\Delta t,\Delta x}u = f$, we say that the finite difference scheme is *consistent* with the partial differential equation if for any smooth function $\phi(x,t)$

$$P\phi - P_{\Delta t,\Delta x}\phi \to 0 \text{ as } \Delta t, \Delta x \to 0.$$

Definition 2.19:

A finite difference scheme $P_{\Delta t,\Delta x} v_i^n = 0$ for a first-order equation is *stable* in a stability region Λ if there is an integer *J* such that for any positive time *T*, there is a constant C_T such that

$$\left\| v^n \right\|_{\Delta x} \le C_T \sum_{j=0}^{J} \left\| v^j \right\|_{\Delta x}$$

for $0 < n < T$, with $(\Delta t, \Delta x) \in \Lambda$.

Definition 2.20: (trapezoidal rule)

The *trapezoidal rule* is an approximate technique for calculating the definite integral, the formula is given by

$$\int_a^b p(x)dx \approx (b-a)\frac{p(b)+p(a)}{2}$$

Definition 2.21: (Averages of functions)

The average of a given function on a interval [*a, b*] is given by

$$\overline{p} = \frac{1}{b-a}\int_a^b p(x)dx$$

Remark: Using the trapezoidal rule we have $\overline{p} = p$.

Definition 2.22: (Consistence)

A given scheme is precise to order l in time and space if the error between the exact and the approximate solutions is $O(\delta t^l)$ and $O(\delta x^l)$.

We say that the scheme is *p*-consistence by taking p to be the minimum of the precision in time and in space.

Definition 2.23: (Stability)

Let V be the vector space, p^n is said to be stable in V if

$\exists C_T > 0 \left\| p^n \right\|_V \leq C_T$ with $t = n\delta t \leq T$ and $\delta t \to 0$.

Theorem 2.7: (Lax theorem)

If the problem well posed and the approximation scheme is consistence the stability is equivalent to convergence.

2.7 MATHEMATICAL MODELS

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. Mathematical models are used in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (such as computer science, artificial intelligence), as well as in the social sciences (such as economics, psychology, sociology, political science). Physicists, engineers, statisticians, operations research analysts, and economists use mathematical models most extensively. A model may help to explain a system and to study the effects of different components, and to make predictions about behavior. Mathematical models can take many forms, including dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed.

A *mathematical models* can be broadly defined as set of equation that expresses the essential features of a physical system in terms of variables that describe the system. The mathematical models of physical phenomena are often based on fundamental scientific laws of physics such as the principle of conservation of mass, conservation of linear momentum, and conservation of energy. Below we consider three simple examples drawn from dynamics, heat transfer and solid mechanics to illustrate how mathematical models of physical problem are formulated. In the physical sciences, the traditional mathematical model contains four major elements. These are

(1) Governing equations

(2) Defining equations

(3) Constitutive equations

(4) Constraints

Mathematical models are usually composed of relationships and variables. Relationships can be described by operators, such as algebraic operators, functions, differential operators, etc. Variables are abstractions of system parameters of interest, that can be quantified. Several classification criteria can be used for mathematical models according to their structure:

- **Linear vs. nonlinear:** If all the operators in a mathematical model exhibit linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise. The definition of linearity and nonlinearity is dependent on context, and linear models may have nonlinear expressions in them. For example, in a statistical linear model, it is assumed that a relationship is linear in the parameters, but it may be nonlinear in the predictor variables. Similarly, a differential equation is said to be linear if it can be written with linear differential operators, but it can still have nonlinear expressions in it. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is regarded as a linear model. If one or more of the objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.

Nonlinearity, even in fairly simple systems, is often associated with phenomena such as chaos and irreversibility. Although there are exceptions, nonlinear systems and models tend to be more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one is trying to study aspects such as irreversibility, which are strongly tied to nonlinearity.

- **Static vs. dynamic:** A dynamic model accounts for time-dependent changes in the state of the system, while a static (or steady-state) model calculates the system in equilibrium, and thus is time-invariant. Dynamic models typically are represented by differential equations or difference equations.

- **Explicit vs. implicit:** If all of the input parameters of the overall model are known, and the output parameters can be calculated by a finite series of computations, the model is said to be explicit. But sometimes it is the output parameters which are known, and the corresponding inputs must be solved for by an iterative procedure, such as Newton's method (if the model is linear) or Broyden's method (if non-linear). In such a case the model is said to be implicit. For example, a jet engine's physical properties such as turbine and nozzle throat areas can be explicitly calculated for a given design thermodynamic cycle (air and fuel flow rates, pressures, and temperatures) at a specific flight condition and power setting, but the engine's operating cycles at other flight conditions and power settings cannot be explicitly calculated from the constant physical properties.

- **Discrete vs. continuous:** A discrete model treats objects as discrete, such as the particles in a molecular model or the states in a statistical model; while a continuous model represents the objects in a continuous manner, such as the velocity field of fluid in pipe flows, temperatures and stresses in a solid, and electric field that applies continuously over the entire model due to a point charge.

- **Deterministic vs. probabilistic (stochastic):** A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables; therefore, a deterministic model always performs the same way for a given set of initial conditions. Conversely, in a stochastic model—usually called a "statistical model"—randomness is present, and variable states are not described by unique values, but rather by probability distributions.

- **Deductive, inductive, or floating:** A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation, but is merely the invocation of expected structure. Application of mathematics in social sciences outside of economics has been criticized for unfounded models. Application of catastrophe theory in science has been characterized as a floating model.

2.8 BOUNDARY VALUE, INITIAL VALUE AND EIGENVALUE PROBLEMS

The objective of most analyses is to determinant unknown functions, called *dependent variables*, that are governed by a set of differential equations posed in a given domain Ω and some conditions on the boundary Γ of the domain. Often, a domain not including its boundary is called an open domain. A domain Ω with its boundary Γ is called a closed domain and is denoted by $\overline{\Omega} = \Omega \cap \Gamma$.

When the dependent variables are functions of one independent variable (say, *x*), the domain is a linear segment (i.e., one dimension) and the end points of the domain are called boundary points. If the domain is two dimensional and the boundary is the closed curve enclosed it. And in three dimensional domain, the boundary is two surface.

A differential equation is said to describe a *boundary value problem* over the domain Ω if the dependent variable and possibly its derivative are required to take specified values on the boundary Γ of Ω. An *initial value problem* is one which the dependent variable and possibly its derivative are specified initially (i.e., at time $t = 0$). A problem can be both a boundary value and initial value problem if the dependent variable is subject to both boundary and initial conditions. Another type of problem we encounter is one in which a differential equation governing the dependent unknown also contains an unknown parameter and we are required to find both the dependent variable and the parameter such that the differential equation and associated boundary conditions are satisfied. Such problems are called *eigenvalue problems*. Examples of varies types of problems in science and engineering are given below.

2.8.1 Boundary Value Problems

Bending of Elastic Beams under Transverse Load: Find $u(x)$ that satisfies the second-order differential equation and *boundary conditions*:

$$\frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right)+cu=f \text{ for } 0<x<L$$

$$u(0)=u_0, \quad \left(\frac{du}{dx}\right)_{x=0}=d_0$$

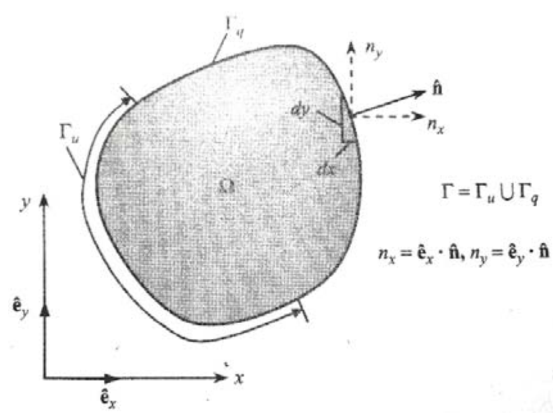


Fig.2.2: Two-dimensional domain

$$\left[\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)\right]_{x=L}=m_0, \quad \left(b\frac{d^2u}{dx^2}\right)_{x=L}=v_0$$

Steady Heat Conduction in a Two-Dimensional Region and Transverse Deflections of a Membrane (Fig.2.2): Find $u(x, y)$ that satisfies the second-order partial differential equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+cu=f \text{ in } \Omega$$

$$u=u_0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=q_0 \text{ on } \Gamma_q$$

where (n_x, n_y) are the direction cosines of the unit normal vector $\hat{\boldsymbol{n}}$ to the boundary Γ_q.

2.8.2 Initial Value Problems

A general second-order equation: Find $u(t)$ that satisfies the second-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+b\frac{d^2u}{dt^2}+cu=f \quad \text{for } 0<t\le T$$

$$u(0)=u_0, \quad \left(b\frac{du}{dt}\right)_{t=0}=v_0$$

2.8.3 Boundary and Initial Value Problems

Unsteady Heat Transfer in a Rod: Find $u(x, t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\rho\frac{\partial u}{\partial t}=f(x,t) \quad \text{for } 0<x<L, 0<t<T$$

$$u(0,t)=d_0(t), \quad \left(a\frac{\partial u}{\partial x}\right)_{x=L}=q_0(t), \quad u(x,0)=u_0(x)$$

Unsteady Motion of a Membrane: Find $u(x, y, t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+\rho\frac{\partial^2 u}{\partial t^2}=f(x,y,t) \text{ in } \Omega$$

$$u=u_0(t) \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=q_0(t) \text{ on } \Gamma_q$$

$$u(x,y,0)=d_0, \quad \dot{u}(x,y,0)=v_0$$

where the superposed dot indicates a derivative with respect to t.

2.8.4 Eigenvalue Problems

Transverse Vibrations of a Membrane: Find $u(x, y)$ and λ that satisfy the differential equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]-\lambda u=0 \text{ in } \Omega$$

$$u=0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=0 \text{ on } \Gamma_q$$

The value of λ are called *eigenvalues*, and the associated functions u are called *eigen-functions*.

The set of specified functions and parameters (e.g., *a, b, c, f,* u_0, d_0, q_0, v_0 and so on) are called the data of the problem. Differential equations in which the right-hand side f is zero are called *homogeneous differential equations,* and boundary (initial) conditions in which the specified data is zero are called homogeneous boundary (initial) conditions. The *exact solution* of a differential equation is the function that identically satisfies the differential equation at every point of the domain and for all times $t > 0$, and satisfies the specified boundary and/or initial conditions.

2.9 FINITE VOLUME METHOD

The finite volume method (FVM) is a method for representing and evaluating partial differential equations in the form of algebraic equations (LeVeque, 2002; Toro, 1999), expressing the conservation or balance of one or more quantities. These partial differential equations are often called conservation laws; they may be of different nature, e.g. elliptic, parabolic or hyperbolic, and they are used as models in a wide number of fields, including physics, biophysics, chemistry, image processing, finance, dynamic reliability. It describes the relations between partial derivatives of unknown fields such as temperature, concentration, pressure, molar fraction, density of electrons or probability density function, with respect to variables within the domain (space, time, etc.) under consideration. Similar to the finite difference method or finite element method, it has been extensively used in several engineering fields, such as fluid mechanics, heat and mass transfer or petroleum engineering. In FVM, the values are calculated at discrete places on a meshed geometry. 'Finite volume' refers to the small volume surrounding each node point on a mesh. In the finite volume method, volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, these methods are conservative. Another advantage of the finite volume method is that it is easily formulated to allow for unstructured meshes. The method is used in many computational fluid dynamics packages (Toro, 1999; LeVeque, 2002).

Among the some of the features of the finite volume method, its application in the arbitrary geometries using structured or unstructured meshes, and it leads to robust schemes, are most prominent. An additional feature is the local conservativeity of the numerical fluxes that is the numerical flux is conserved from one discretization cell to its neighbor. This last feature makes the finite volume method quite attractive when modelling problems for which the flux is of importance, such as in fluid mechanics, semi-conductor device simulation, heat and mass transfer, etc. It is said that the finite volume method is locally conservative because it is based on a 'balance' approach: a local balance is written on each discretization cell which is often called 'control volume'. Moreover, by applying the divergence formula, an integral formulation of the fluxes over the boundary of the control volume is obtained. The fluxes on the boundary are discretized with respect to the discrete unknowns.

However, the finite volume method is quite different from the finite difference method or the finite element method. Each method is quite similar in that it represents a systematic numerical method for solving partial differential equations. One important difference is the ease of implementation. A common opinion is that the finite difference method is the easiest to implement and the finite element method the most difficult. One reason for this may be that the finite volume method requires quite sophisticated mathematics for its formulation. The finite volume method's strength is that it only needs to do flux evaluation for the cell boundaries. This also holds for nonlinear problems, which makes it extra powerful for robust handling of (nonlinear) conservation laws appearing in transport problems.

The local accuracy of the finite-volume method, such as close to a corner of interest, can be increased by refining the mesh around that corner, similar to the finite element method. However, the functions that approximate the solution when using the finite volume method cannot be easily made of higher order. This is a disadvantage of the finite volume method compared to the finite element and finite difference methods. For example, the finite difference method becomes difficult to use when the coefficients involved in the equation are discontinuous (e.g. in the case of heterogeneous media). With the finite volume method, discontinuities of the coefficients will not be any problem if the mesh is chosen such that the discontinuities of the coefficients occur on the boundaries of the control volumes (for elliptic problems). Thus, the finite volume scheme differs from the finite difference scheme in that the finite difference approximation is used for the flux rather than for the operator itself (Eymard et al. 2006).

2.10 FINITE ELEMENT MODELS

At present, the Finite Element procedures are widely using in the engineering analysis, and can be expect that this method will find its wider applications in the years to come. This procedure is employed extensively in the analysis of solids and structures, heat transfer and fluids, and indeed, considered that the finite element methods are useful in virtually every field of engineering analysis.

The development of finite element methods for the solution of practical engineering problems began with the advent of the digital computer. That is, the essence of a finite element solution of an engineering problem is that a set of governing algebraic equations is established and solved, and it was only through the use of the digital computer that this process could be rendered effective and given general applicability. These two properties, effectiveness and general applicability, in engineering analysis-are inherent and have been developed to a high degree for practical computations, so that finite element methods have found wide appeal in engineering practice.

As is often the case with original developments, it is rather difficult to quote an exact 'date of invention', but the roots of the finite element method can be traced back to three separate research groups: applied mathematicians, physicists and engineers. Although in principle published already, the finite element method obtained its real impetus from the developments of engineers. Since the early 1960s, a large amount of research has been devoted to the technique, and a very large number of publications on the finite element method is available.

The finite element method in engineering was initially developed on a physical basis for the analysis of problems in structural mechanics. However, it was soon recognized that the technique could be applied equally well to the solution of many other classes of problems. The finite element method is employed to solve very complex mathematical models, but it is important to realize that the finite element solution can be never give more information that contained in the mathematical model.

The finite element method is used to solve physical problems in engineering analysis and design. Fig.2.3 summarizees the process of finite element analysis. The physical problem typically involves an actual structure or structural component subjected to certain assumption that together lead to differential equations governing the mathematical model. Since the finite element solution technique is a numerical procedure, it is necessary to assess the solution accuracy. If the accuracy criteria are not met, the numerical (i.e. finite element) solution has to be repeated with refined solution parameters(such as finer meshes) until a sufficient accuracy is reached (Bathe, 1996).

Any numerical simulation, such as the finite element method, is not an end in itself but rather an aid to design and manufactureing. There are several reasons why an engineer or a scientist should study a numerical method, especially the finite element method (Reddy, 2006):

(1) Most practical problems involve complicated domains (both geometry and material constitution), loads and nonlinearities that forbid the devlopment of analytical solutions. Therefore, the only alternative is to find approximate solutions using numerical methods.

(2) A numerical method, with the advent of a computer, can be used to investigate the effects of various parameters (e.g., geometry, material parameters, or loads) of the system on its response to gain a better understanding of the process/system being analyzed. It is cost-effective and saves time and material resources compared to the multitude of physical experiments needed to gain the same level of understanding.

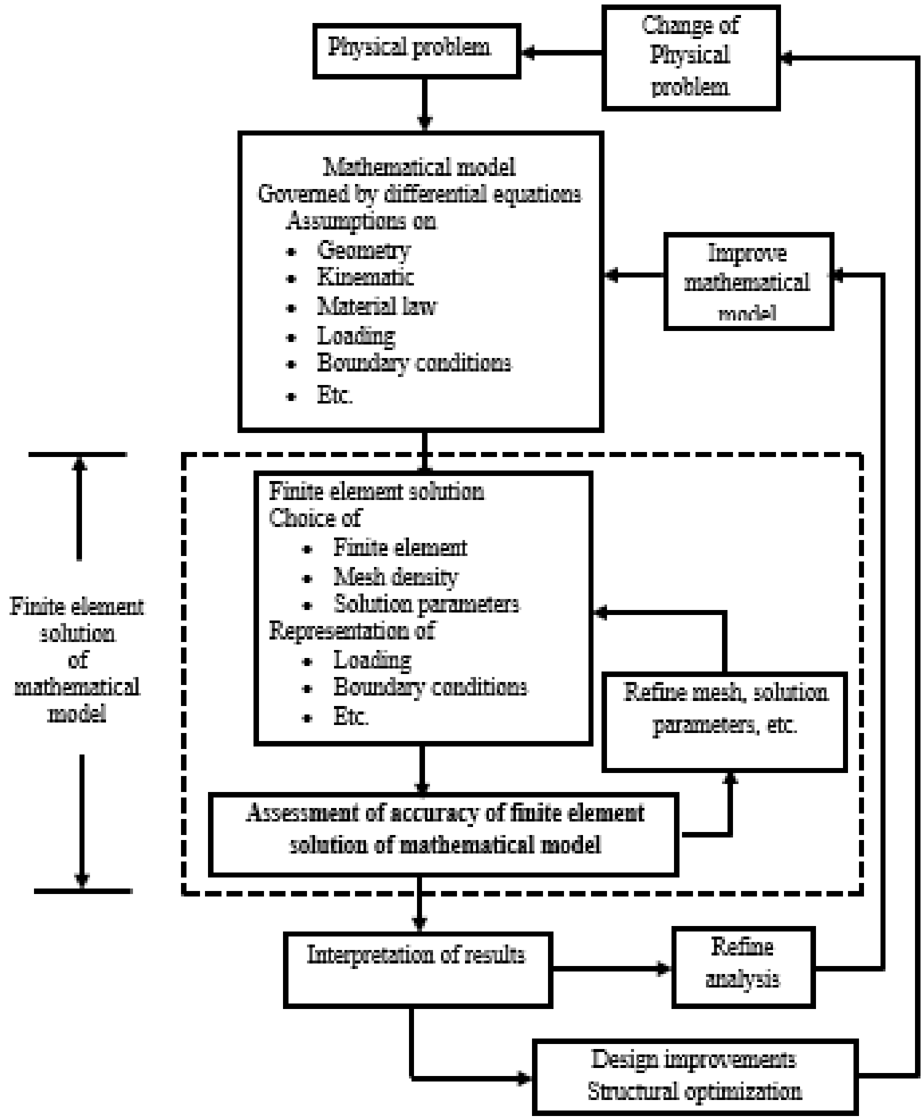


Fig.2.3: The process of finite element analysis.

(3) Because of the power of numerical methods and electronic computation, it is possible to include all relevant features in a mathematical model of a physical process without worrying about its solution by exact means.

(4) Those who are quick to use a computer program rather than *think* about the problem to be analyzed may find it difficult to interpret or explain the computer-generated results. Even to develop proper input data to the computer program. A good understanding of the underlying theory of

the problem as well as the numerical methods (on which the computer program is based) is required.

(5) The finite element method and its generalizations are the most powerful computer oriented methods ever devised to analyze practiacl engineering problems. Today, finite element analysis is an integral and major component in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace, chemical, pharmaceutical, petroleum, electronice, and communications, as well as emerging technologiies such as nanotechnology and biotechnology rely on the finite element method to simulate complex phenomenon at scales for design and manufacture of high-technology products.

Many introductory texts on the finite element method discuss the solution for linear problems of elasticity and field equations. In practical applications, the limitation of linear elasticity, or more generally of linear behavior, often precludes obtaining an accurate assessment of the solution because of the presence of 'nonlinear' effects and/or because the geometry has a 'thin' dimension in one or more directions. Zienkiewicz and Taylor (2005) describe extensions to the formulations introduced to solve linear problems to permit solutions to both classes of problems.

Nonlinear behavior of solids takes two forms: material non-linearity and geometric non-linearity. The simplest form of non-linear material behavior is that of elasticity for which the stress is not linearly proportional to the strain. More general situations are those in which the loading and unloading response of the material is different. Typical example here is the case of classical elastic–plastic behavior.

When the deformation of a solid reaches a state for which the undeformed and deformed shapes are substantially different a state of finite deformation occurs. In this case it is no longer possible to write linear strain–displacement or equilibrium equations on the undeformed geometry. Even before finite deformation exists it is possible to observe buckling or load bifurcations in some solids and non-linear equilibrium effects need to be considered. The classical Euler column, where the equilibrium equation for buckling includes the effect of axial loading, is an example of this class of problem. When deformation is large the boundary conditions can also become nonlinear. Examples are pressure loading that remains normal to the deformed body and also the case where the deformed boundary interacts with another body. This latter example defines a class known as contact problems and much research is currently performed in this area. An example of a class of problems involving non-linear effects in deformation measures, material behaviour and contact is the analysis of a rolling tyre. A typical mesh for a tyre analysis is shown in Fig.2.4. The cross-section shown is able to model the layering of rubber and cords and the overall character of a tyre. The full mesh is generated by sweeping the cross-section around the wheel axis with a variable spacing in the area which will be in contact. A formulation in which the mesh is fixed and the material rotates is commonly used to perform the analysis (Zienkiewicz and Taylor, 2005).

Generally the accurate solution of solid problems which have one (or more) small dimension(s) compared to the others cannot be achieved efficiently using standard two- or three-dimensional finite element formulations. Traditionally separate theories of structural mechanics are introduced to solve this class of problems. A plate is a flat structure with one thin (small) direction which is called the thickness. A shell is a curved structure in space with one such small thickness direction. Structures with two small dimensions are called beams, frames, or rods. A primary reason why use of standard two- or three-dimensional finite element formulations do not yield accurate solutions is the numerical ill-conditioning which results in their algebraic equations. Zienkiewicz and Taylor (2005) combines the traditional approaches of structural mechanics with a much stronger link to the full three-dimensional theory of solids to obtain formulations which are easily solved using standard finite element approaches.

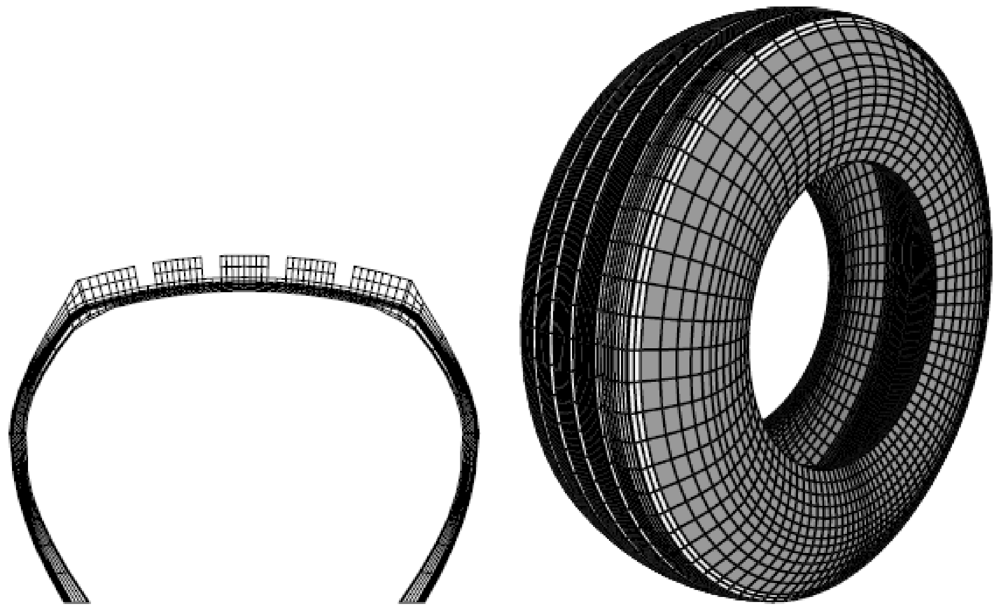


Fig.2.4: Finite element mesh for tyre analysis.

2.10.1 The Model Equation

Consider the problem of finding the solution u of the second-order partial differential equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}u-f=0 \qquad (2.12)$$

For given data $a_{ij}\left(i,j=1,2\right), a_{00}$ and f, and specified boundary condition. The form of boundary conditions will be apparent from the weighted-integral (Galerkin weighted residual method) formulation. As a special case, one can obtain the Poisson equation from Eq. (2.12) by setting $a_{11}=a_{22}=a$ and $a_{12}=a_{21}=a_{00}=0$; we get

$$\vec{\nabla}.\left(a\vec{\nabla}u\right)=f \text{ in } \Omega \qquad (2.13)$$

where $\vec{\nabla}$ is the gradient operator.

As in the following, one can develop the finite element model of Eq.(2.12). The major steps as follows:

(1) Discretization of the domain into a set of finite elements.
(2) Weighted-integral (Galerkin weighted residual method) formulation of the governing differential equation.
(3) Derivation of finite element interpolation functions.
(4) Development of the finite element model using the Galerkin approach.
(5) Assembly of finite elements to obtain the global system of algebraic equations.
(6) Imposition of boundary conditions.
(7) Solution of the equations.
(8) Post-computation of the solution and quantities of interest.

2.10.2 Finite Element Discretization

The general rules of mesh generation for finite element formulations include the following:

(1) Select elements that characterize the governing equations of the problem.
(2) The number, shape and type (i.e. or linear, quadratic) elements should be such that the geometry of the domain is represented as accurately as desired.
(3) The density of elements should be such that regions of large gradients of the solution are adequately modeled (i.e., more or higher-order elements should be used in regions of large gradients).
(4) Mesh refinements should very gradually from higher-density regions to lower- density regions. If *transition elements* are used, they should be used away from critical regions (i.e., regions of large gradients). Transition elements are those that connect lower-order elements to higher-order elements (e.g., linear to quadratic).

2.10.3 General form of Some Physical Problems

The general field equation

$$D_x \frac{\partial^2 \varphi}{\partial x^2} + D_y \frac{\partial^2 \varphi}{\partial y^2} - G\varphi + Q = 0 \qquad (2.14)$$

has many important applications in the physical sciences. A few of these equations are discussed in this section.

(1) The first application area is the torsion of noncircular sections. With $Dx = Dy = \frac{1}{g}, G = 0, Q = 2\theta$, the field equation reduces to the torsion problem

$$\frac{1}{g}\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{g}\frac{\partial^2 \varphi}{\partial y^2} + 2\theta = 0 \qquad (2.15)$$

where g is the shear modulus of the material and θ is the angle of twist.

(2) Several fluid mechanics problems are embedded within the field equation. The streamline and potential formulations for an ideal irrotational fluid are governed by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (2.16)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad (2.17)$$

where the streamline ψ, are perpendicular to the constant potential lines φ, and the velocity components are related to the derivatives of either φ or ψ with respect to x and y.

(3) The seepage of water under a dam or retaining wall and within a confined aquifer is governed by

$$D_x \frac{\partial^2 \varphi}{\partial x^2} + D_y \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (2.18)$$

where D_x and D_y are the permeabilities of the earth material and φ represents the piezometric head.

(4) The water level around a well during the pumping process is governed by

$$D_x \frac{\partial^2 \varphi}{\partial x^2} + D_y \frac{\partial^2 \varphi}{\partial y^2} + Q = 0 \qquad (2.19)$$

where Q is a point sink term, D_x and D_y are the permeabilities of the earth material and φ represents the piezometric head and aquifer is assumed to be confined.

(5) The heat transfer from a two-dimensional fin to the surrounding fluid by convection is governed by

$$D_x \frac{\partial^2 \varphi}{\partial x^2} + D_y \frac{\partial^2 \varphi}{\partial y^2} - \frac{2h}{t} T + \frac{2hT_h}{t} = 0 \qquad (2.20)$$

the coefficients D_x and D_y represent the thermal conductivities in the x and y directions, respectively; h is the convection coefficient; t is the thickness of the fin; T_f is the ambient temperature of the surrounding fluid; and T is the temperature of the fin. If the fin is assumed to be thin and the heat loss from the edges is neglected.

When the body is very long in the z-direction and the temperature is a function of only the x- and y-direction, the heat transfer is governed by

$$Dx \frac{\partial^2 T}{\partial x^2} + Dy \frac{\partial^2 T}{\partial y^2} = 0 \qquad (2.21)$$

(6) Seiche motion, which describes the standing waves on a boundary shallow body of water, is governed by

$$h \frac{\partial^2 w}{\partial x^2} + h \frac{\partial^2 w}{\partial y^2} + \frac{4\pi^2}{gT^2} = 0 \qquad (2.22)$$

where h is the water depth at the quiescent state, w is the wave height above the quiescent level, g is the gravitational constant, and T is the period of oscillation.

(7) A fluid vibrating within closed volume is governed by

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{w^2}{c^2} P = 0 \qquad (2.23)$$

where P is the pressure excess above ambient pressure, w is the wave frequency, and c is the wave velocity in the medium.

Above different physical problems are contained within the general differential equation. Finite element formulation of these equation is given in detail by Segerlind (1984).

2.10.3 Finite Element Formulations of Field Problems

For clarity and reference, we consider some field problems (Torsion problem, Elasticity problem, Heat conduction problem, Parabolic equations, Hyperbolic equations) to present finite element formulations of element equations.

2.10.3.1 Torsion Problem (Galerkin's Approach)

Usually, the torsion problem, with simply connected cross- sections is written as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + h = 0 \quad \text{in } \mathbf{A} \tag{2.24}$$

$$\phi = 0 \qquad on \qquad C^*$$

$$\frac{\partial \phi}{\partial \eta} = 0 \qquad on \qquad C$$

where C*, C constitute the cross section boundaries and $\mathrm{h} = 2\mathrm{g}\theta$.

Using of weighted residuals $\iint_{\Omega} WRd\Omega = 0$

where
$$R = \frac{\partial^2 \phi}{\partial^2 x} + \frac{\partial^2 \phi}{\partial^2 y} + h \tag{2.25}$$

and W is the weighting function.

After using Green's theorem, boundary conditions and using shape functions as

$$\varphi = \sum_{j=1}^{NP} N_j \varphi_j \tag{2.26}$$

where N_j are the shape functions.

In Galerkin formulation we set $W_j = N_j$,

Now the Eq.(2.24) becomes

$$\sum_{j=1}^{NP}\left[-\iint_{\Omega}\left(\frac{\partial W}{\partial x}\frac{\partial \varphi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial \varphi}{\partial y}\right)d\Omega\right]\varphi_j = \iint_{\Omega} hN_i d\Omega, \quad 1\le i\le m \qquad (2.27)$$

where NP is the number of nodes in the domain Ω.

It is clear that Eq.(2.27) can be written as

$$[K]\{\phi\} = \{F\}$$

where the components of [K] and { F} are given by

$$\left.\begin{aligned} K_{ij} &= \iint_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)dxdy \\ F_i &= \iint_{\Omega} hN_j dxdy \end{aligned}\right\} \qquad (2.28)$$

Matrix [K] usually known as the stiffness matrix which is symmetric and { F} is called the load vector. It is clear that the evaluation of the stiffness matrix [K] requires to integrate the product of the global derivatives of shape functions.

2.10.3.2 Elasticity Problem (Galerkin's Approach)

Two-dimensional elasticity is generally categorized into two modes: plane strain and plane stress. When the thickness of a solid object is large, a state of plane strain is considered to exist. If this thickness is small compared to its overall dimensions (x,y), the condition of plane stress is assumed. Both cases are subsets of general three-dimensional elasticity problems. In this instance, body forces (or loads) cannot have components in the z-direction, nor vary in the direction of the body thickness. Fig.2.5 illustrates plane stress and plane strain the governing equations (details can be seen in Reddy 1984, 2006) which described two-dimensional elastic stress are defined as

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+f_x = 0 \qquad (2.29)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \tag{2.30}$$

where σ_x and σ_y are the normal stress components in the *x*- and *y*-directions, respectively; τ_{xy} is the shear stress which acts in the *x*-*y* plane; and f_x and f_y are the body force terms. The strain-displacement relations are defined from the Eq.(2.29) as

$$-\frac{\partial}{\partial x}\left(C_{11}\frac{\partial u}{\partial x} + C2\frac{\partial v}{\partial y}\right) - C_{33}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = f_x$$

and from the Eq.(2.30) as

$$-C_{33}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(C_{12}\frac{\partial u}{\partial x} + C_{22}\frac{\partial v}{\partial y}\right) = f_y$$

with the boundary tractions given by

$$t_x = \left(C_{11}\frac{\partial u}{\partial x} + C_{22}\frac{\partial v}{\partial y}\right)\eta_x - C_{33}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\eta_y$$

$$t_y = C_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\eta_x + \left(C_{12}\frac{\partial u}{\partial x} + C_{22}\frac{\partial v}{\partial y}\right)\eta_y$$

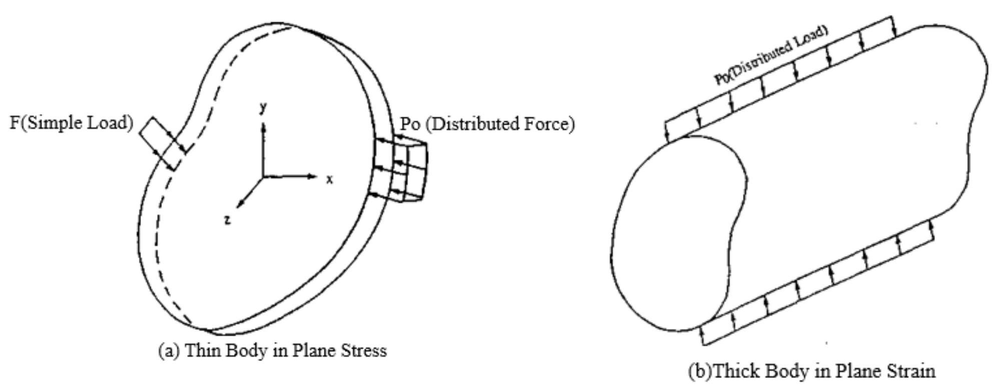


Fig.2.5: Plane stress and plane strain.

After simplification as of Eq.(2.24), we have from the Eq.(2.29)

$$\sum_{j=1}^{n} \iint_A u_j \left(C_{11} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + C_{33} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

$$+\sum \iint_A v_j \left(C_{12} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + C_{33} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} \right) dxdy$$

$$= \iint_A N_i f_x dxdy + \int_{S_1} N_i t_x ds \qquad i = 1,2,3,..........,n$$

and from the Eq.(2.30), we have

$$\sum_{j=1}^{n} \iint_A u_j \left(C_{33} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + C_{12} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} \right) dxdy$$

$$+\sum \iint_A v_j \left(C_{33} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + C_{22} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

$$= \iint_A N_i f_y dxdy + \int_{S_2} N_i t_y ds \qquad i = 1,2,3,..........n$$

Both the equations can be written in matrix form as

$$K_{11} u + K_{12} v = F_1$$

$$K_{21} u + K_{22} v = F_2$$

where

$$\left.\begin{aligned}
K_{11} &= \iint_A \left(C_{11} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + C_{33} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy \\
K_{12} = K_{21}^T &= \iint_A \left(C_{12} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + C_{33} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} \right) dxdy \\
K_{22} &= \iint_A \left(C_{33} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + C_{22} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy \\
F_1 &= \iint_A N_i f_x dxdy + \int_{S_1} N_i t_x ds \\
F_2 &= \iint_A N_i f_y dxdy + \int_{S_2} N_i t_y ds
\end{aligned}\right\} \qquad (2.31)$$

for $i, j = 1,2,3.........NP$.

NP is the node specified over the element of the domain A.

2.10.3.3 Heat Conduction problem (Galerkin's Approach)

The application of the finite element method to two-dimensional problems has only been described in general terms. Now a particular two-dimensional problem is chosen, and the process of determining the coefficients of the corresponding element matrices is described in detail. The problem to be considered is that of solving the familiar equation for steady-state heat conduction in two dimensional,

$$k_x \frac{\partial^2 \phi}{\partial^2 x} + k_y \frac{\partial^2 \phi}{\partial^2 y} + Q = 0 \quad \text{in } \Omega \tag{2.32}$$

where k_x and k_y , are the thermal conductivities and Q is an internal heat source or sink.

The differential Eq.(2.32) is embedded within Eq.(2.14). The parameters for Eq.(2.14) are

$$D_x = k_x, \qquad D_y = k_y, \qquad G = 0, \qquad Q = Q$$

with standard boundary conditions

$$\phi = \bar{\phi} \qquad on \quad \Gamma_\phi$$

$$k \frac{\partial \phi}{\partial n} = -\bar{q} \qquad on \quad \Gamma_q$$

where $\Gamma_\phi + \Gamma_q = \Gamma$.

Now the weighted residual statement, using the trial functions themselves as weighting functions, can then be deduced as the similar procedure in the previous cases, similarly we obtain,

$$\int_\Omega \left(\frac{\partial N_l}{\partial x} k_x \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial N_l}{\partial y} k_y \frac{\partial \hat{\phi}}{\partial y} \right) dxdy = \int_\Omega Q N_l dxdy - \int_{\Gamma_q} \bar{q} N_l d\Gamma$$

This Equation can be written in matrix form as

$$K\phi = f$$

in which the components of the matrices K and f are determined by summing the individual element contributions

$$K_{lm}^{e} = \int_{\Omega} \left(k_x \frac{\partial N_l^e}{\partial x} \frac{\partial N_m^e}{\partial x} + k_y \frac{\partial N_l^e}{\partial y} \frac{\partial N_m^e}{\partial y} \right) \; dxdy$$

$$f_l^e = \int_{\Omega^e} Q N_l^e dxdy - \int_{\Gamma_q^e} \bar{q} N_l^e d\Gamma$$

Here Ω^e is the surface of the element e and Γ_q^e is the portion of the element boundary which lies on, or approximates to, a portion of Γ_q . This finite element formulation is completely general. We wish to note that one can use only the general triangular or the convex quadrilateral elements as well as both type of elements if necessary for descretization of the problem domain. At a glance, it can be said that in Finite Element equation of the physical problems following integrals are needed to evaluate to form the element matrices:

$$\left.\begin{array}{l} \iint\limits_A \dfrac{\partial N_i}{\partial x}\dfrac{\partial N_j}{\partial x} dxdy \\ \iint\limits_A \dfrac{\partial N_i}{\partial y}\dfrac{\partial N_j}{\partial y} dxdy \\ \iint\limits_A \dfrac{\partial N_i}{\partial x}\dfrac{\partial N_j}{\partial y} dxdy \\ \iint\limits_A \dfrac{\partial N_i}{\partial y}\dfrac{\partial N_j}{\partial x} dxdy \\ \iint\limits_A N_i dxdy \\ \int\limits_{s_1} N_i ds \\ \int\limits_{s_2} N_i ds \end{array}\right\} \qquad (2.33)$$

for *i, j=1,2,3……….NP*

NP is the node specified over the element and s_1, s_2 part of the boundary of the domain A.

2.10.3.4 Parabolic Equations

Consider the partial differential equation governing the transient heat transfer and like problems in a two dimensional region Ω with both boundary Γ ,

$$c\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)+a_0u=f(x,y,t) \qquad (2.34)$$

with the boundary conditions

$$u=\hat{u} \quad \text{or} \quad q_n=\hat{q}_n \text{ on } \Gamma \quad (t\geq 0)$$

where

$$q_n=a_{11}\frac{\partial u}{\partial x}n_x+a_{22}\frac{\partial u}{\partial y}n_y$$

The initial conditions (i.e., at t = 0) are of the form

$$u(x,y,0)=u_0(x,y) \text{ in } \Omega$$

Here t denotes time and $c, a_{11}, a_{22}, a_0, \hat{u}, f$ and $\hat{q}_n$ are given functions of position and/or time. Eq.(2.34) is a modification of Eq.(2.12) in that it contains a time derivative term, which accounts for time variations of the physical process represented by Eq.(2.12).

Now the weighted residual statement, using the trial functions themselves as weighting functions, can then be deduced as the similar procedure in the previous cases, similarly we obtain in matrix form

$$[M^e]\{\dot{u}^e\}+[K^e]\{u^e\}=\{f^e\}+\{Q^e\}$$

where a superposed dot on u denotes a derivative with time ($\dot{u}=\partial u/\partial t$) and

$$M_{ij}^e=\int_{\Omega_e}cN_i^eN_j^e dxdy$$

$$K_{ij}^e=\int_{\Omega_e}\left(a_{11}\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x}+a_{22}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}+a_0N_i^eN_j^e\right)dxdy$$

$$f_i^e=\int_{\Omega_e}f(x,y,t)N_i^e dxdy$$

Details derivation of the above equations (can be seen in Reddy, 1984, 2006) from which it is clear that the following integrals are needed to evaluate to form the element matrices:

$$\left.\begin{array}{l}\iint_A \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}dxdy,\ \iint_A \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}dxdy,\ \iint_A \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}dxdy \\ \iint_A \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}dxdy,\ \iint_A N_i N_j dxdy,\ \int_{\Omega_e} N_i ds\end{array}\right\} \tag{2.35}$$

for $i, j=1,2,3\ldots\ldots\ldots NP$

NP is the node specified over the element of the domain A.

2.10.3.5 Hyperbolic Equations

The transverse motion of a membrane, for example, is governed by the partial differential equation of the form,

$$c\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)+a_0 u = f(x,y,t) \tag{2.36}$$

where $u(x, y, t)$ denotes the transverse deflection, c the material density of the membrane, with the boundary conditions a_{11} and a_{22} are the tensions in the x and y directions of the membrane, a_0 is the modulus of elastic foundation on which the membrane is stretched (often $a_0=0$ i.e., there is no foundation), and $f(x, y, t)$ is the transversely distributed force. Eq.(2.36) is known as the *wave equation* and classified mathematically an a hyperbolic equation. The function u must be determined such that it satisfies Eq. (2.36) in a region Ω and the following boundary and initial conditions:

$$u=\hat{u} \quad \text{or} \quad q_n=\hat{q}_n \ \text{ on } \Gamma \quad (t\geq 0)$$

$$u(x,y,0)=u_0(x,y), \quad \frac{\partial u}{\partial t}(x,y,0)=v_0(x,y)$$

where $\hat{u}$ and $\hat{q}_n$ are specified boundary values of u and q_n, and u_0 and v_0 are specified initial values of u and its derivative , respectively.

Following the similar procedure of the previous cases, we obtain in matrix form

$$[M^e]\{\ddot{u}^e\}+[K^e]\{u^e\}=\{f^e\}+\{Q^e\}$$

For the components M_{ij}^e, K_{ij}^e and f_i^e we are required to evaluate integrals same as for previous parabolic equation.

Chapter 3

Development of a Mathematical Model for Better Management of Fisheries Resources

Chapter 3

Development of a Mathematical Model for Better Management of Fisheries Resources

This chapter describes the size and space-time structured model of fish population dynamics. It is expressed as a hyperbolic system of partial differential equations continuous in size and time with non-local boundary data. The space is represented using discrete regions (patches) interconnected by exchange rates. The model is nonlinear because all the parameters such as birth rate, growth function, mortality rate, and movements are chosen to be size-dependent and age independent. This article provides the mathematical model and proves existence and uniqueness of the solution. By using the finite volume method the continuous problem is discretized and then upwind explicit scheme is developed. Consequently, this model approaches the problem by the upwind explicit scheme where the consistence and stability are established. A computer code in programming language FORTRAN® is developed and appended in Appendix I, compatible with the formulation is used to compute the numerical solutions and the results are shown graphically.

3.1 INTRODUCTION

Fish and fishery products are one of the most widely traded agricultural commodities with exports worth. Fisheries not only meets the demand of the necessary animal protein consumed globally to feed an ever-increasing human population, but also provides an employment for more than two billion people worldwide. This consumption of food fish is increasing as the world population is increasing geometrically, especially in many countries in Asia, Africa and South America. The fisheries are not been able to keep pace with the growing demand, while fisheries in some developed countries are recovering, overfishing is impoverished the state of the ecosystem globally. This threat extensively demands the establishment of sustainable fisheries to explore the challenges of increasing demand of fishmeal for human population.

One of the major tasks for fisheries owners is to attempt to regulate their fishery in such a way as to obtain the maximum benefit from it. Therefore the owner has to have some options of restricting or encouraging fishing effort, setting catch quotas, and restricting the legal size of the fish captured, and should attempt to control these in such a way as to ensure the fishery meets his management objectives. As the fishery is a complex system and it is not easy to interpret the wide range of data that can be obtained about such diverse features as growth rates, behaviour with respect to fishing gear, fertility, etc., nor is it easy to predict the effect on the fishery of changes, such as increasing the minimum size at which fish may be taken, or decreasing the fishing effort (Allen, 1975).

Therefore, several mathematical models are presented for the study of fish population. Zhang et al. (2000) proposed and analyzed a model to study the optimal harvesting policy of a stage structured problem and derived necessary and sufficient condition for the coexistence and extinction of species. Song and Chen (2001) deliberated the optimal harvesting policy and stability for a two-species competitive system and derived the conditions for the existence of a globally asymptotically stable positive equilibrium and a threshold of harvesting for the mature population. Dubey et al. (2002) presented a dynamic model for a single-species fishery which depends partially on a logistically growing resource. They showed that both the equilibrium density of fish population as well as the maximum sustainable yield increases as the resource biomass density increases. Faugeras and Maury (2005a) described a multi-region nonlinear age-size structured fish population model to assess its mathematical posedness based on initial boundary value problem and existence and uniqueness of a positive weak solution is proved.

Fish population dynamics describes the ways in which a given population grows and shrinks over time, as controlled by birth, death, emigration or immigration and fishing. Dubey et al. (2003) proposed and analyzed a mathematical model to study the dynamics of a fishery resource system using the Pantryagin's maximum principle. Faugeras and Maury (2005b) developed an advection-diffusion size-structured fish population dynamics model to simulate the skipjack tuna population in the Indian Ocean. The model is fully spatialized, and movements are parameterised with oceanographical and biological data. Van Kooten et al. (2010) developed a mathematical model to study the relation between hatching size and response to harvesting mortality. The result shows that the hatching size determines dynamics through its effect on the relative strength of cannibalistic mortality. Kar (2006) proposed and analysed a nonlinear mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists of two zones; a free fishing zone and a reserve zone where fishing is strictly prohibited. He observed that in the absence of predator, even under continuous harvesting, fish population may be maintained at an appropriate equilibrium level. Most of these models are setup to preserve the fish species disappearing, to provide assessment of the fish abundance and fishery exploitation in order to determine sustainable yield such that economic purposes or ecological yield (Wentworth et al., 2011). However, most of the models are of weakly coupled hyperbolic partial differential equations with nonlocal boundary conditions (Aylaj and Noussair, 2010).

Moreover, dynamic models for the commercial fishing taking into account the economic and ecological factors have been studied extensively. Link et al. (2011) studied the role of the fishermen's harvesting strategies based on the economic impacts of changes in fish population dynamics. This study shows that the fishing strategy is based on a short optimisation period of only two fishing periods, changes in population dynamics have a direct influence on the returns from fishing due to the strong pressure on the stocks applied by the fisheries. If the strategy is based on a longer optimisation period, fishing activities may be deferred to allow for stock regrowth, which improves the economic performance of the fisheries. However, in that case the relationship between population dynamics and fishing activities becomes less clear. Carson et al. (2009) studied the classical Gordon-Schaefer fishery management model by replacing the constant growth rate with a cyclical growth rate. In this study the optimal harvest rate is shown to fluctuate, but the cycle of the harvest rate lags the cycle of the biological growth function with the highest harvest rate occurring after biological conditions start to decline. They also showed that small cyclical fluctuations in one species can result in large fluctuations in the optimal harvest rate of another species if the fish species are interlinked through predator-prey relationships.

The dynamic model approach, therefore, is a widely applied suitable technique to a number of environmental management and sustainability issues particularly to the fisheries management problems (Bendor et al., 2009; Martinet et al., 2010; De Lara et al., 2011). Later on, Dubey and Patra (2013) developed a dynamic model for optimal management and utilisation of a renewable resource by population. An appropriate Hamiltonian function is formed for the discussion of optimal harvesting of the resources. Mansal et al. (2014) modelled the time evolution of the resources, the fishing effort and the price which is assumed to vary with respect to supply and demand. Solving the variability problems for nonlinear dynamic system relied on the consistency between a controlled dynamic and acceptability constraints applying both to states and decisions of the system.

From the aforementioned literatures it is found that the birth (or recruitment) and mortality rate, harvesting , emigration or immigration rate, setting catch quotas, restricting the legal size of the fish capture, etc. are the controlled parameters of fish population growth. In this work, a mathematical model which is useful for the different fish species has been developed. Indeed, most fish population share specific characteristics, which have to be taken into account in order to model their dynamics in a realistic manner. In this model the population dynamics in which both size and time are taken as structure variables to account for growth, mortality, movements of fish, environmental variability and variable distribution of fishing effort have been considered as input variables within the multi-regions. This model is established considering a system of hyperbolic partial differential equations where both linear and nonlinearities boundary conditions are discussed. The second and most important goal is to assess the mathematical and numerical analysis of the model to compute the solution numerically.

3.2 MATHEMATICAL ANALYSIS OF THE MODEL

In this chapter, a hypothetical mathematical model of the fish population is carried out as a trial problem and as a result the hyperbolic system of partial differential equations (PDE) is observed. Considering the aforestated parameters the existence and the uniqueness of the continuous problem have been solved.

3.2.1 Derivation of the model

Let the geographic domain Ω, which contains the sub-domains Ω_k such that

$\Omega = \bigcup_{1\le k\le nk} \Omega_k$, where *nk* is the number of regions, shown in Fig.3.1.

Again let P be the total fish-population in the domain, Ω. The number of individuals of length between 0 and *L*, in the domain Ω_k at time *t* is given by

$$P^k(t) = \int_0^L p^k(x,t)dx$$

where $p^k(x,t)$ is the density of fish population of length *x* at time *t* in the zones Ω_k . $x \in (0,L)$ and $t \in (0,T)$ are continuous variables with $L,T \in \mathbb{R}$ .Setting $\mathfrak{D} = (0,L)\times(0,T)$.

The basic model of population dynamics of fisheries is

$$P^k(t+\delta t) = P^k(t) + \text{birth rate} + \text{migration rate} - \text{mortality rate} - \text{emigration rate} \tag{3.1}$$

where $P^k(t+\delta t)$ is the number of fish at time $t+\delta t, \delta t$ is the time variation. The mortality rate includes both the fishing mortality and natural mortality.

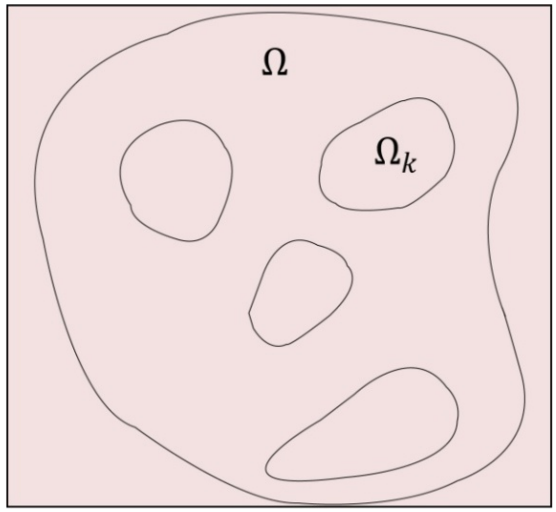


Fig. 3.1: Geographic domain, Ω and sub-domains, Ω_k

At time *t*, in the domain, Ω_k, and using the following notations

$\mu_k = \mu_k(x,t),$ the natural morality

$h_k = h_k(x,t),$ the harvest morality

$\beta_k = \beta_k(x,t),$ the birth (recruitment)

$v^k = v^k(x,t),$ growth function

m_{k-l} the migration rate of individuals going from Ω_k to the region Ω_l.

m_{l-k} the number that emigrated from Ω_l to the region Ω_k.

where μ_k, h_k, v^k, β_k are all non-negative functions of the size *x* and time *t* , and m_{k-l} and m_{l-k} are non-negative constant functions.

At the time *t*, the length of new and the old individuals are increasing with a growth function v^k, this is represented by $v^k p^k(x,t), v^k p^k(x-\delta x,t)$ respectively, and as the time goes on i.e., at $(t+\delta t)$ the density of the new and old fish is given by $p^k(x,t)\delta x$ and $p^k(x,t+\delta t)\delta x$. The fish population of same age sometime have different size, this may due to the genetic or the environment.

Using the notation and the processes that we defined above, the Eq.(3.1) becomes

$$\begin{cases} p^k(x,t+\delta t)\delta x = p^k(x,t)\delta x + v^k(x+\delta x,t)p^k(x+\delta x,t)\delta t - v^k(x,t)p^k(x,t)\delta t \\ -\mu^k(x,t)p^k(x,t)\delta x\delta t - h^k(x,t)p^k(x,t)\delta x\delta t - \left(\sum_{k\neq l} m_{k\to l}\right)p^k(x,t)\delta x\delta t \\ +\left(\sum_{l\neq k} m_{l\to k}\right)p^k(x,t)\delta x\delta t, \qquad (x,t)\in\mathfrak{D} \end{cases} \tag{3.2}$$

The birth rate or recruitment, will be inside the boundary condition. By dividing the Eq.(3.2) by $\delta x\delta t$, one obtains

$$\begin{cases} \dfrac{p^k(x,t+\delta t)-p^k(x,t)}{\delta t} = \dfrac{v^k(x+\delta x,t)p^k(x+\delta x,t)-v^k(x,t)p^k(x,t)}{x} \\ -\left(\mu^k(x,t)p^k(x,t)\delta x\delta t - h^k(x,t)p^k(x,t)\right) - \left(\sum_{k\neq l} m_{k\to l}\right)p^k(x,t) \\ +\left(\sum_{l\neq k} m_{l\to k}\right)p^k(x,t), \qquad (x,t)\in\mathfrak{D} \end{cases} \tag{3.3}$$

Passing to the limit as δx and δt tend to 0, the hyperbolic system of partial differential equations (PDEs) with *nk* unknowns

$$\mathbf{p}(x,t) = \left(p^1(x,t), p^2(x,t), \cdots, p^k(x,t)\right), \quad 1 \le k \le nk$$

$$\begin{cases} \dfrac{\partial p^k(x,t)}{\partial t} + \dfrac{\partial (v^k(x,t)p^k(x,t))}{\partial x} = -\mu^k(x,t)p^k(x,t) - h^k(x,t)p^k(x,t) \\ \qquad -\left(\sum_{k\neq l} m_{k\to l}\right)p^k(x,t) + \left(\sum_{l\neq k} m_{l\to k}\right)p^k(x,t) \qquad (x,t)\in\mathfrak{D} \end{cases} \tag{3.4}$$

We shall complete this equation with an initial and a boundary conditions.

The initial density associates to initial growth function of fish population is given by

$$p^k(x,0) = p_0^k, \quad x \in (0,L) \tag{3.5}$$

The boundary condition is defined by

$$v^k(0,t)p^k(0,t) = \int_0^L \beta(x,t)p^k(x,t)dx, \quad t \in (0,T) \tag{3.6}$$

We define the matrix M of migration and emigration rate as follow

$$M_{kl} = \begin{cases} m_{l\to k} & \text{if } k \neq l \\ -\sum_{i\neq k}^{nk} m_{k\to i} & \text{if } k = l \end{cases}$$

The explicit form of the matrix M is written

$$M = \begin{pmatrix} -\sum_{i\neq 1}^{nk} m_{1\to i} & m_{2\to 1} & m_{3\to 1} & \cdots & m_{nk\to 1} \\ m_{1\to 2} & -\sum_{i\neq 2}^{nk} m_{2\to i} & m_{3\to 2} & \cdots & m_{nk\to 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1\to nk} & m_{2\to nk} & m_{3\to nk} & \cdots & -\sum_{i\neq k}^{nk} m_{k\to i} \end{pmatrix}$$

Using definition of the matrix M, we set

$$\sum_{l\neq k} m_{l\to k} p^l(x,t) - \sum_{k\neq l} m_{k\to l} p^k(x,t) = (Mp)^k \tag{3.7}$$

where

$$(M_p)^k = \begin{pmatrix} -\sum_{i\neq 1}^{nk} m_{1\to i} & m_{2\to 1} & m_{3\to 1} & \cdots & m_{nk\to 1} \\ m_{1\to 2} & -\sum_{i\neq 2}^{nk} m_{2\to i} & m_{3\to 2} & \cdots & m_{nk\to 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1\to nk} & m_{2\to nk} & m_{3\to nk} & \cdots & -\sum_{i\neq k}^{nk} m_{k\to i} \end{pmatrix} \begin{pmatrix} p^1 \\ p \\ \vdots \\ p^{nk} \end{pmatrix}$$

By replacing Eq.(3.7) into Eq.(3.4) we obtain

$$\begin{cases} \dfrac{\partial p^k(x,t)}{\partial t} + \dfrac{\partial (v^k(x,t)p^k(x,t))}{\partial x} = -g^k(x,t)p^k(x,t) - (Mp)^k, \quad (x,t) \in \mathfrak{D} \\ \qquad p^k(x,0) = p_0^x, \qquad x \in (0,L) \\ \qquad v^k(0,t)p^k(0,t) = \int_0^L \beta(x,t)p^k(x,t)dx, \quad t \in (0,T) \end{cases} \tag{3.8}$$

3.2.2 Existence and uniqueness

To have a good regularity of the solution it is convenient to make these suppositions.

Hypothesis 1: For all $k \in [0, nk]$ we make this following assumption:

- v^k and $\partial_x v^k \in L^\infty(\mathfrak{D})$
- $\mu^k \in L^\infty(\mathfrak{D})$
- $h^k \in L^\infty(\mathfrak{D})$ we also set $g^k = h^k + \mu^k$
- $\beta^k \in L^\infty(\mathfrak{D})$
- $m_{l\to m}$ and $m_{k\to l} \in L^\infty(\mathfrak{D})$
- $p_0^x \in L^2(\mathfrak{D})$

Hypothesis 2: Furthermore, we assume the $\partial_x v^k$ and $g^k = h^k + \mu^k$ are non-negative, Lipschitz and continuous with respect to the size *x* and the time *t*.

***Step 1*:**

In order to prove existence and uniqueness it is convenient to make a change of variable.

Here, setting $p^k(x,t) = \exp(\lambda t)\tilde{p}^k(x,t)$, and replacing into Eq.(3.8), the equation becomes

$$\begin{cases} \dfrac{\partial \tilde{p}^k(x,t)}{\partial t} + \dfrac{\partial (v^k(x,t)\tilde{p}^k(x,t))}{\partial x} = -(g^k(x,t)+\lambda)\tilde{p}^k(x,t) - (M\tilde{p})^k, \quad (x,t)\in \mathfrak{D} \\ \quad \tilde{p}^k(x,0) = \tilde{p}_0^x, \qquad x\in(0,L) \\ \quad v^k(0,t)\tilde{p}^k(0,t) = \int_0^L \beta(x,t)\tilde{p}^k(x,t)dx, \quad t\in(0,T) \end{cases}$$

In the remained of the work, our change of unknown is implicit, we will use p^k instead of $\tilde{p}^k$.

The proof of existence and uniqueness consists of two main steps. First of all, we show the result in the case that the boundary condition is independent of the fish density and then we use ***fixed point theorem*** to prove the result with the original boundary condition.

Let f^k be fixed in $L^2(\mathfrak{D})$, we introduce the problem

$$\begin{cases} \dfrac{\partial p^k(x,t)}{\partial t} + \dfrac{\partial (v^k(x,t)p^k(x,t))}{\partial x} = -(g^k(x,t)+\lambda)p^k(x,t) - (Mp)^k, \quad (x,t)\in \mathfrak{D} \\ \quad p^k(x,0) = p_0^x, \qquad x\in(0,L) \\ \quad v^k(0,t)p^k(0,t) = \int_0^L \beta(x,t)f^k(x,t)dx, \quad t\in(0,T) \end{cases}$$

***Step 2*:**

Lemma 1: We rewrite the above equation as follows

$$\begin{cases} \dfrac{\partial p^k(x,t)}{\partial t} + v^k(x,t)\dfrac{\partial p^k(x,t)}{\partial x} = -\left(\dfrac{\partial v^k(x,t)}{\partial x} + g^k(x,t)+\lambda\right)p^k(x,t) \\ \qquad\qquad - (Mp)^k, \quad (x,t)\in \mathfrak{D} \\ p^k(x,0) = p_0^k, \qquad x\in(0,L) \\ v^k(0,t)p^k(0,t) = \int_0^L \beta(x,t)f^k(x,t)dx, \quad t\in(0,T) \end{cases} \tag{3.9}$$

The problem thus defined has a unique solution.

Proof: Using characteristic method for first order of partial differential equations. To do that, we are required to seek for a characteristic curve ($x(s)$,$t(s)$) on which the partial differential Eq.(3.9), becomes an ordinary differential equation. By using chain rule, we get

$$\frac{\partial p^k(x(s),t(s))}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial p^k}{\partial x} + \frac{\partial t}{\partial s}\frac{\partial p^k}{\partial t}$$

Setting, $\frac{\partial x(s)}{\partial s} = v^k, \frac{\partial t(s)}{\partial s} = 1.$

Therefore,

$$\frac{\partial p^k}{\partial s} = \frac{\partial p^k}{\partial t} + v^k(x,t)\frac{\partial p^k}{\partial x} = -\left(\frac{\partial v^k(x,t)}{\partial x} + g^k(x,t) + \lambda\right)p^k(x,t) - (Mp)^k$$

Hence, along the characteristic curve ($x(s)$; $t(s)$), the original PDE becomes the ODE

$$\frac{\partial p^k}{\partial s} = -\left(\frac{\partial v^k(x,t)}{\partial x} + g^k(x,t) + \lambda\right)p^k(x,t) + \left(Mp\right)^k$$

So, we have to solve the characteric system of ODE

$$\begin{cases} \dfrac{\partial t(s)}{\partial s} = 1, t(0) = 0 \\ \dfrac{\partial x(s)}{\partial s} = v^k, x(0) = x_0 \\ \dfrac{dp^k}{ds} = -\left(\dfrac{\partial v^k(x,t)}{\partial x} + g^k(x,t) + \lambda\right)p^k(x,t) + \left(Mp\right)^k, p^k(0) = p_0^k(x_0) \end{cases}$$

The solutions are given by

(1). $\frac{\partial t(s)}{\partial s} = 1, \forall s \in (0,T), t(s) = s$

(2). $\frac{\partial x(s)}{\partial s} = v^k, \forall s \in (0,T), x = v^k s + x_0, x = v^k t + x_0$

Let us write the third equation in the matrix form

$$\frac{d\mathbf{p}}{ds} = \mathbf{A}\mathbf{p}$$

where $\mathbf{p}(x,t) = (p^1(x,t), p^2(x,t), \cdots, p^{nk}(x,t))^T$ and

$$\mathbf{A} = \begin{pmatrix} -\gamma^k - \sum_{i\neq 1}^{nk} m_{1\to i} & m_{2\to 1} & m_{3\to 1} & \cdots & m_{nk\to 1} \\ m_{1\to 2} & -\gamma^k - \sum_{i\neq 2}^{nk} m_{1\to i} & m_{3\to 2} & \cdots & m_{nk\to 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1\to nk} & m_{2\to nk} & m_{3\to nk} & \cdots & -\gamma^k - \sum_{i\neq k}^{nk} m_{k\to i} \end{pmatrix}$$

with $\gamma^k = \dfrac{\partial v^k(x,t)}{\partial x} + g^k(x,t) + \lambda.$

According to the hypothesis 2, γ^k is continuous and Lipchitz, since $m_{k\to l}$, $m_{l\to k}$ are constants for $1 \le k, l \le nk$ it follows that the matrix **A** is continuous and Lipchitz, by Cauchy-Lipschitz theorem the equation $\dfrac{d\mathbf{p}}{ds} = \mathbf{A}\mathbf{p}$ has unique solution which is $\mathbf{p}(x(s), t(s)) = p_0(x - v^k t)\exp\left(\int_0^T \mathbf{A}dt\right).$

Thus the existence and uniqness of the solution of Eq.(3.9) hold.

***Step 3*:**

Now introduce the following application

$$F : L^2(\mathfrak{D}) \to L^2(\mathfrak{D})$$

$$f^k \mapsto p^k$$

Lemma 2: The application F is constructed *i.e.*

$$\exists C > 0 \in [0,1] \text{ such that, } \left\| F(f_2^k) - F(f_1^k) \right\|_{L^2} \le C \left\| f_2^k - f_1^k \right\|$$

Proof: To prove this lemma, we shall use the priori estimate. Let p_1^k and p_2^k be two solutions of Eq.(3.9), therefore, the difference between $p_1^k - p_2^k$ need to verify. Now, from Eq.(3.9)

$$\begin{cases} \dfrac{\partial(p_2^k - p_1^k)}{\partial t} + \dfrac{\partial v^k(x,t)(p_2^k - p_1^k)}{\partial x} = -\left(g^k(x,t) + \lambda\right)(p_2^k - p_1^k) \\ \qquad\qquad\qquad\qquad\qquad + \left((Mp_2)^k - (Mp_1)^k\right) \\ (p_2^k(x,0) - p_1^k(x,0)) = (p_{0,2}^k - p_{0,1}^k) \\ v^k(0,t)(p_2^k(0,t) - p_1^k(0,t)) = \int_0^L \beta(x,t)(f_2^k - f_1^k)dx \end{cases}$$

Setting $w^k = p_1^k - p_2^k$, we get

$$\begin{cases} \dfrac{\partial w^k}{\partial t} + \dfrac{\partial v^k(x,t)w^k}{\partial x} = -\left(g^k(x,t) + \lambda\right)w^k + \left((Mp_2)^k - (Mp_1)^k\right) \\ w^k(x,0) = w_0^k \\ v^k(0,t)w^k(0,t) = \int_0^L \beta(x,t)(f_2^k - f_1^k)dx \end{cases} \tag{3.10}$$

By multiplying the Eq.(3.10) by w^k, and by integrating on $\mathfrak{D} = (0,L)\times(0,T)$, we obtain

$$\begin{aligned} \int_{\mathfrak{D}} \frac{1}{2}\partial_t (w^k)^2 dtdx + \int_{\mathfrak{D}} \frac{1}{2}\partial_x (w^k)^2 dtdx = &-\int_{\mathfrak{D}} (g^k + \lambda)(w^k)^2 dtdx \\ &+ \int_{\mathfrak{D}} w^k \left((Mp_2)^k - (Mp_1)^k\right)dtdx \end{aligned} \tag{3.11}$$

Integrating the first term on (0, *T*) and using Definition 2.7, we get

$$\begin{aligned} \int_{\mathfrak{D}} \frac{1}{2}\partial_t (w^k)^2 dtdx &= \int_{\mathfrak{D}} \frac{1}{2}\left((w^k(x,T))^2 - (w^k(x,0))^2\right)dtdx - \int_{\mathfrak{D}} \frac{1}{2}(w^k)^2 dtdx \\ &= -\frac{1}{2}\left\|w_0^k\right\|_{L^2(0,L)}^2 - \frac{1}{2}\left\|w^k\right\|_{L^2(\mathfrak{D})}^2 + \int_0^L \frac{1}{2}(w^k(x,T))^2 dx \end{aligned}$$

we assume that $v^k(0,t) \neq 0$, we integrate by part the second term, we find

$$\int_{\mathfrak{D}} \frac{1}{2}\partial_t v^k (w^k)^2 dtdx = -\int_{\mathfrak{D}} \frac{1}{2} v^k \partial_x (w^k)^2 dt + \int_0^T \left(v^k(L,t)(w^k(L,t))^2 - v^k(0,t)(w^k(0,t))^2\right)dt$$

$$
\begin{aligned}
&= -\int_{\mathfrak{D}} v^k \partial_x (w^k)^2 \, dt dx + \int_0^T v^k(L,t)(w^k(L,t))^2 dt \\
&- \int_0^T \frac{1}{v^k(0,t)} dx \int_0^L \ (\beta(x,t))^2 (f_2^k - f_1^k) dx dt
\end{aligned} \tag{3.12}
$$

We replace into Eq.(3.11) and using Definition 2.8, we have

$$
-\frac{1}{2}\left\|w_0^k\right\|^2_{L^2(0,L)}\left(\lambda-\frac{1}{2}\right)-\frac{1}{2}\left\|v^k\right\|_\infty \int_0^T \left[(w^k)^2\right]_0^L dt+\left\|v^k\right\|_\infty \left\|w^k(L,t)\right\|^2_{L^2(0,L)}
$$

$$
\left\|w^k\right\|^2_{L^2(\mathfrak{D})} \le \frac{\left\|\beta^2\right\|_\infty}{(\lambda+\left\|v^k\right\|_\infty - nk^2\left\|M\right\|_\infty - 1)\left\|v^k\right\|_\infty}\left\|p_2^k - p_1^k\right\|^2_{L^2(\mathfrak{D})} \le \left\|f_2^k - f_1^k\right\|^2_{L^2(\mathfrak{D})}
$$

After simplification, we find

$$
\left\|w^k\right\|^2_{L^2(\mathfrak{D})} \le \frac{\left\|\beta^2\right\|_\infty}{(\lambda+\left\|v^k\right\|_\infty - nk^2\left\|M\right\|_\infty - 1)\left\|v^k\right\|_\infty}\left\|f_2^k - f_1^k\right\|^2_{L^2(\mathfrak{D})}
$$

Therefore,

$$
\left\|p_2^k - p_1^k\right\|^2_{L^2(\mathfrak{D})} \le \left\|f_2^k - f_1^k\right\|^2_{L^2(\mathfrak{D})}
$$

Hence, F is a contraction for

$$
C = \frac{\left\|\beta^2\right\|_\infty}{(\lambda+\left\|v^k\right\|_\infty - nk^2\left\|M\right\|_\infty - 1)\left\|v^k\right\|_\infty}
$$

and $$\lambda > \left\|v^k\right\|_\infty - nk\left\|M\right\|_\infty - 1$$

Thus, according to the **Banach fixed point theorem**, F admits a unique fixed point p^k, which is the desired solution; so existence and uniqueness of the Eq.(3.8) hold.

3.3 NUMERICAL APPROXIMATION OF THE MODEL

The goal of this section is to approximate the solution of the hyperbolic system of partial differential equation given in previous section using finite volume method.

3.3.1 Upwind explicit scheme

Recalling the continuous problem Eq.(3.8)

$$\begin{cases} \dfrac{\partial p^k(x,t)}{\partial t} + \dfrac{\partial (v^k(x,t)p^k(x,t))}{\partial x} = -g^k(x,t)p^k(x,t) - (Mp)^k, \quad (x,t) \in \mathfrak{D} \\ p^k(x,0) = p_0^x, \qquad x \in (0,L) \\ v^k(0,t)p^k(0,t) = \int_0^L \beta(x,t)p^k(x,t)dx, \quad t \in (0,T) \end{cases}$$

In order to approach correctly a given hyperbolic problem, especially the transport equation, and seeking information from where it comes, it states that if the coefficient of the convection term is positive the propagation is to be taken place at the right. Therefore, it is clear that if the coefficient term is negative, the propagation is to be taken place at the left. In this model, the coefficient v^k which is a non-negative function, hence the propagation will be at the right.

To establish the scheme, we define a mesh of rectangles in the *xt*-plane call control volume as shown in Fig.3.2 is defined. Letting $\delta x = \dfrac{L}{(nx+1)}$ and $\delta t = \dfrac{T}{(nt+1)}$, where δx the steps and *nx* the number of nodes in *x* axes, δt step in time. Again, define $x_i = i\delta x$ and $t_n = n\delta t, 1 \le n \le nt, 1 \le i \le nl$. Inside each rectangle, we are defining $p^k = p^k(x_i,t_n) = p_i^{k,n}$, where $p^k(x_i,t_n)$ is the approximate solution of the exact solution at time t_n and local size x_i.

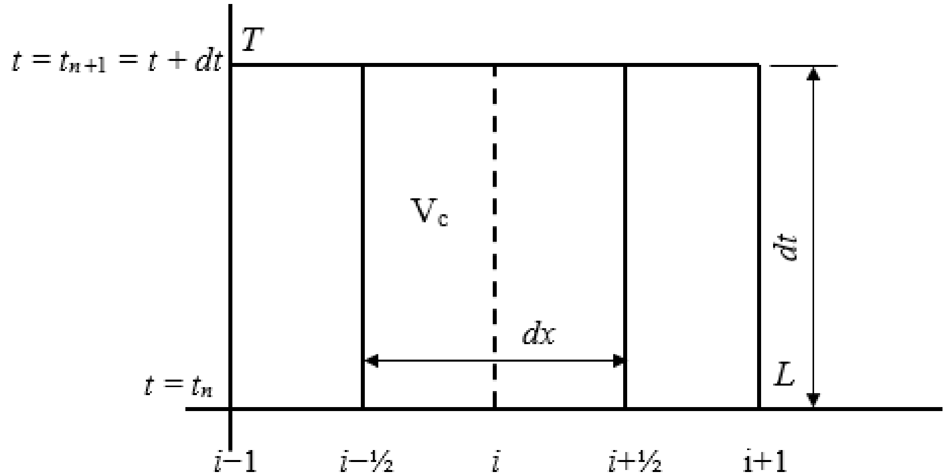


Fig. 3.2: Schematic representation of the control volume, called vertex

Integrating the continuous problem on the control volume with respect to the time and space,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t}^{t+\delta t} \partial_t p^k(x,t)dtdx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t}^{t+\delta t} \partial_t v^k(x,t)p^k(x,t)dtdx$$

$$= -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t}^{t+\delta t} g^k(x,t)p^k(x,t)dtdx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t}^{t+\delta t} (Mp)^k dtdx$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[p^k(x,t)\right]_t^{t+\delta t} dx + \int_{t}^{t+\delta t} \left[v^k(x,t)p^k(x,t)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$

$$= -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t}^{t+\delta t} g^k(x,t)p^k(x,t)dtdx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t}^{t+\delta t} (Mp)^k dtdx$$

Using trapezoidal rule and choosing the value at time $n+\frac{1}{2}$ to the left, it obtains

$$\begin{aligned}&\delta x\left(p_i^{k,n+1} - p_i^{k,n}\right) + \delta t\left(v_{i+1/2}^{k,n+1/2}p_{i+1/2}^{k,n+1/2} - v_{i-1/2}^{k,n+1/2}p_{i-1/2}^{k,n+1/2}\right)\\ &= -\delta x\delta t\ g_i^{k,n+1/2}p_i^{k,n+1/2} + \delta x\delta t(Mp)_i^{k,n+1/2}\end{aligned} \tag{3.13}$$

assume $n+\frac{1}{2}\approx n, i+\frac{1}{2}\approx i, i-\frac{1}{2}\approx i-1$ by dividing Eq.(3.13) by $\delta x\delta t$ we get the following explicit scheme:

$$\begin{cases} \dfrac{p_i^{k,n+1}-p_i^{k,n}}{\delta t}+\dfrac{v_i^{k,n}p_i^{k,n}-v_{i-1}^{k,n}p_{i-1}^{k,n}}{\delta x}=-g_i^{k,n}p_i^{k,n}+(Mp)_i^{k,n} \\ v_0^{k,n}p_0^{k,n}=\sum_{i=1}^{nl}\beta_i^{k,n}p_i^{k,n}dx \end{cases} \tag{3.14}$$

where

$$p_i^{k,n}=p^k(x,t),\ p_{i-1}^{k,n}=p^k(x-\delta x,t),\ v_{i-1}^{k,n}=v^k(x-\delta x,t),$$
$$p_i^{k,n+1}=p^k(x,t+\delta t),(Mp)_i^{k,n}=(Mp)^k(x,t) \text{ and } g_i^{k,n}=g^k(x,t)$$

3.3.2 Consistency, order of accuracy

The consistency and the order of accuracy of the Euler explicit scheme defined in Eq.(3.14):

$$L_{ap}u:\frac{p_i^{k,n+1}-p_i^{k,n}}{\delta t}+\frac{v_i^{k,n}p_i^{k,n}-v_{i-1}^{k,n}p_{i-1}^{k,n}}{\delta x}=-g_i^{k,n}p_i^{k,n}+(Mp)_i^{k,n}$$

and Lu the continuous problem given in Eq.(3.8).

Using Taylor expansion of p^k around $t+\delta t$, we get

$$p^k(x,t+\delta t)=p^k(x,t)-\delta x\frac{\partial}{\partial x}\left(p^k(x,t)\right)+\mathrm{O}(\delta x^2)$$

Hence

$$\frac{p^k(x,t+\delta x)-p^k(x,t)}{\delta t}=\frac{\partial}{\partial t}\left(p^k(x,t)\right)+\mathrm{O}(\delta t) \tag{3.15}$$

Similarly, expanding p^k and v^k around $x+\delta x$,

$$p^k(x-\delta x,t)=p^k(x,t)-\delta x\frac{\partial}{\partial x}\left(p^k(x,t)\right)+\mathrm{O}(\delta x^2) \tag{3.16}$$

$$v^k(x-\delta x,t) = v^k(x,t) - \delta x \frac{\partial}{\partial x}\left(v^k(x,t)\right) + \mathrm{O}(\delta x^2) \tag{3.17}$$

Multiplying Eq.(3.16) by Eq.(3.17),

$$v^k(x-x,t)p^k(x-x,t) = v^k(x,t)p^k(x,t) - \delta x \frac{\partial}{\partial x}\left(v^k(x,t)p^k(x,t)\right) + \mathrm{O}(\delta x^2)$$

Thus

$$\frac{v^k(x,t)p^k(x,t) - v^k(x-\delta x,t)p^k(x-\delta x,t)}{\delta x} = \frac{\partial}{\partial x}\left(v^k(x,t)p^k(x,t)\right) + \mathrm{O}(\delta x) \tag{3.18}$$

Adding Eq.(3.15) and Eq.(3.18), it obtains

$$L_{ap}u - Lu = \mathrm{O}(\delta x) + \mathrm{O}(\delta t) \to 0 \text{ when } \delta x, \delta t \to 0$$

Therefore the explicit Euler scheme is of order 1 in space and of order 1 in time.

3.3.3 Stability

In this section, the L^1 stability of the explicit scheme is studied. The following L^1 discrete norm

$$\left\| p^k \right\|_{L^1} = \sum_{i=1}^{nl} \left| p_i^n \right| \delta x$$

is introduced. In order to prove the stability, the following two lemmas are used:

Lemma 3 :(Stability condition)

Under the condition if $\left(1 - (\frac{\delta t}{\delta x} v_i^{k,n} - \delta t g_i^{k,n})\right) \ge 0$ then $p_i^{k,n+1}$ is positive.

Proof: Let us rewrite the scheme Eq.(3.14) in the following form

$$p_i^{k,n+1} = \left(1 - (\frac{\delta t}{\delta x} v_i^{k,n} - \delta t g_i^{k,n})\right) p_i^{k,n} + \delta t (Mp)_i^{k,n} + \frac{\delta t}{\delta x} v_{i-1}^{k,n} p_{i-1}^{k,n}$$

Since, $v_{i-1}^{k,n} p_{i-1}^{k,n} \ge 0$ and $(Mp)_i^{k,n} \ge 0$ therefore $p_i^{k,n+1}$ is positive if $\left(1 - (\frac{\delta t}{\delta x} v_i^{k,n} - \delta t g_i^{k,n})\right) \ge 0$, thus the lemma hold.

Lemma 4: (Discrete Gromwell's inequality)

Let $p_0^k \in L^2(\mathfrak{D})$, under the positiveness of the scheme, the discrete L^1 -norm satisfies

$$\left\|p^{k,n}\right\|_{L^1} \leq \exp\left(1+\delta t\left(\left(nk\left\|\beta^{k,n}\right\|_{L^\infty}\right)+nk\left\|M\right\|_{L^\infty}\right)\right)\left\|p^{k,0}\right\|_{L^1}$$

Now form Eq.(3.14), we have

$$p_i^{k,n+1} = \left(1-\left(\frac{\delta t}{\delta x}v_i^{k,n} - \delta t g_i^{k,n}\right)\right)p_i^{k,n} + \delta t(Mp)_i^{k,n} + \frac{\delta t}{\delta x}v_{i-1}^{k,n}p_{i-1}^{k,n}$$

By summing the above equation over the indices, i = 1, 2, 3, …, nl, one can obtains the following estimate

$$\sum_{i=1}^{nl} p_i^{k,n+1}\delta x = \sum_{i=1}^{nl}\left(1-\delta t\, g_i^{k,n}\right)p_i^{k,n}\delta x + \frac{\delta t}{\delta x}\sum_{i=1}^{nl}\left(v_{i-1}^{k,n}p_{i-1}^{k,n} - v_i^{k,n}p_i^{k,n}\right)\delta x$$
$$+\sum_{i=1}^{nl}\left(Mp\right)_i^{k,n}\delta x\delta t$$

Using L^1 discrete norm and Definition 2.8 and the matrix M of migration and emigration rate $i.e. M = \sum_{k,l=1}^{nk} M_{k,l}$, we get

$$\left\|p^{k,n+1}\right\|_{L^1} \leq \left\|p^{k,n}\right\|_{L^1} + \delta t\left(v_0^{k,n}p_0^{k,n} - v_{nl}^{k,n}p_{nl}^{k,n}\right) + \sum_{k,n=1}^{nk}\sum_{i=1}^{nl}\left(M_{k,l}p_i^{k,n}\right)\delta x\delta t$$

$$\leq \left\|p_i^{k,n}\right\|_{L^1} + \delta t\int_0^L \beta^{k,n}p_i^{k,n}\delta x + (nk)^2\left\|M\right\|_{L^\infty}\left\|p^{k,n}\right\|_{l^1}\delta t$$

$$\leq \left(1+\delta t\left(\left\|\beta^k\right\|_{L^\infty} + (nk)^2\left\|M\right\|_{L^\infty}\right)\right)\left\|p^{k,n}\right\|_{L^1} \qquad \text{(From Lemma 4)}$$

$$\leq \exp\left(1+\delta t\left\|\beta^k\right\|_{L^\infty} + (nk)^2\left\|M_p\right\|_{L^\infty}\right)\left\|p^{k,0}\right\|_{L^1}$$

So, $p_i^{k,n+1}$ is positive and holds the discrete Gromwell's inequality lemma 4.

Hence this scheme is the stability-L^1. Since our problem is well posed and the upwind explicit scheme is consistent, by the **Lax theorem** (Theorem 2.7) the explicit scheme is convergent.

3.4 NUMERICAL SIMULATION

In order to demonstrate the properties of upwind explicit scheme the numerical experiment has already been defined in the previous sections. Here, for the coding of the scheme the scientific programming language FORTRAN® is used. Therefore, the experiment is realised into two predefined zones, say Ω_1 and Ω_2. Recalling the model equation,

$$\frac{\partial p^k(x,t)}{\partial t}+\frac{\partial (v^k(x,t)p^k(x,t))}{\partial x}=-g^k(x,t)p^k(x,t)-(M_p)^k \quad (x,t)\in\mathfrak{D} \qquad (3.19)$$

Since, the experimental zones are of two sites the Eq.(3.19) becomes

$$\begin{cases}\dfrac{\partial p^1(x,t)}{\partial t}+\dfrac{\partial (v^1(x,t)p^1(x,t))}{\partial x}=-g^1(x,t)p^1(x,t)-m_{12}p^1(x,t)+m_{21}p^2(x,t) \\ \hfill (x,t)\in\mathfrak{D} \\ \dfrac{\partial p^2(x,t)}{\partial t}+\dfrac{\partial (v^2(x,t)p^2(x,t))}{\partial x}=-g^2(x,t)p^2(x,t)+m_{12}p^1(x,t)-m_{21}p^2(x,t) \\ \hfill (x,t)\in\mathfrak{D}\end{cases}$$

Integrate the above system from 0 to L and the new system where unknowns are total population in zones 1 and 2. Using the fact the $v^k(L,t)=0$, k =1,2, the system is written as follows

$$\begin{cases}\dfrac{dP^1(t)}{dt}-v^1(0,t)p^1(0,t)=-\int_0^L g^1(x,t)p^1(x,t)dx \\ \qquad\qquad +\int_0^L\left(m_{21}p^2(x,t)-m_{12}p^1(x,t)\right)dx \\ \dfrac{dP^2(t)}{dt}-v^2(0,t)p^2(0,t)=-\int_0^L g^2(x,t)p^2(x,t)dx \\ \qquad\qquad -\int_0^L\left(m_{21}p^2(x,t)+m_{12}p^1(x,t)\right)dx\end{cases}$$

where $(x,t)\in\mathfrak{D}$

Using the boundary condition,

$$\begin{cases} \dfrac{dP^1(t)}{dt} = \int_0^L \left(\beta^1 - g^1(x,t)\right) p^1(x,t)dx + \int_0^L \left(m_{21}p^2(x,t) - m_{12}p^1(x,t)\right)dx \\ \dfrac{dP^2(t)}{dt} = \int_0^L \left(\beta^2 - g^2(x,t)\right) p^2(x,t)dx - \int_0^L \left(m_{21}p^2(x,t) + m_{12}p^1(x,t)\right)dx \end{cases}$$

where $t \in (0,T)$ (3.20)

Hence, before performing experiments, it is convenient to set the following data:

(1) the maximum time T is one year.

(2) the maximum length L is 90 cm.

The step in time δt is one day and the growth rate $v = 1 = \dfrac{dx}{dt}$, this implies that *dx* is the increment in each day i.e., one centimeter per day.

Considering the fishes are growing 1 cm per day, hence they take 90 days, i.e., three months, to reach the maximum length L. It implies that in every three months of new born, four generations will exist in each year. Therefore, when it reaches to the new generation period, populations in each site increase until three months, and after this period it will start to decrease due to the mortality and/or either fishing or immigrations. This phenomenon is repeated until fish population reach in its equilibrium. Hence, these parameters remain same for the experiments in the linear and nonlinear cases.

3.5 RESULT AND DISCUSSION

3.5.1 The model is linear

In this case all the parameters, the recruitment β^k, harvesting h^k and mortality μ^k are positive constants. If both sites 1 and 2 are protected zones, i.e., no fishing rate, and there is an absence of movement between them, i.e., $m_{12} = m_{21} = 0$, then the system Eq.(3.20) becomes

$$\begin{cases} \dfrac{dP^1(t)}{\partial t} = \left(\beta^1 - \mu^1\right)P^1(t) \quad t \in (0,T) \\ \dfrac{dP^2(t)}{\partial t} = -\left(\beta^2 - \mu^2\right)P^2(t) \quad t \in (0,T) \end{cases} \tag{3.21}$$

Hence, the total population in the two sites are given by

$$P^k(t) = P_0^k \exp((\beta - \mu)t) \tag{3.22}$$

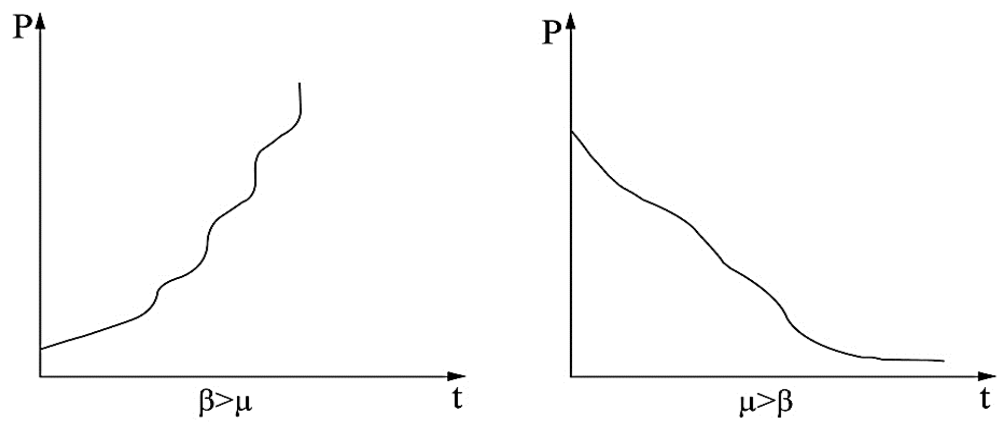


Fig.3.3: Number of total fish population, (i) the rate is increasing exponentially when $\beta > \mu$, and (ii) decreasing when $\beta < \mu$.

But, when $\beta > \mu$, i.e., the number of birth rate is greater than the number of mortality rate, the total fish population in the two regions are increasing exponentially, and it is the opposite i.e., $\beta < \mu$, the population decrease in the same manner as shown in Fig.3.3.

***Case* 1:** For this experiment only one site is under the influence of fishery and the other site is a protected zone. Let the protected zone is site 1, where no harvesting rate, and the fishing area is the region 2, where the harvesting rate is high. It is also assumed that there are no migration and emigration, and both sites 1 and 2 have same natural mortality rate and same initial density. According to these assumptions, setting the following data:

- $\mu_k = 0.03 \quad k = 1,2$
- $h^1 = 0 \quad$ (no fishing in side 1)

- $h^2 = 0.5$ (high harvesting in side 2)
- $m_{21} = m_{12} = 0$ (no migration and emigration in side 1 and 2)

The results of this experiment is shown in Figs.3.4 – 3.6.

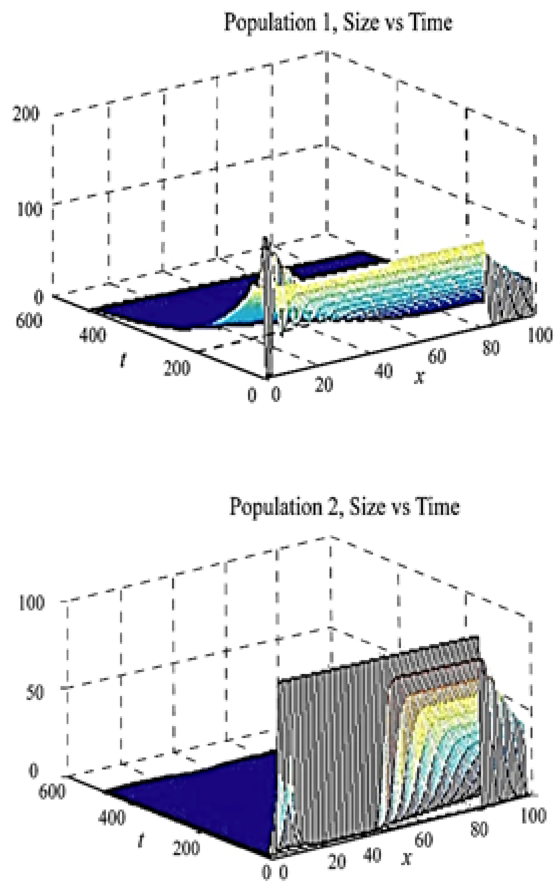
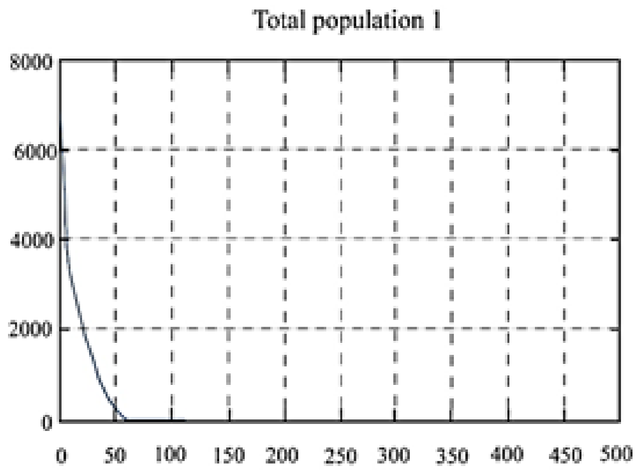


Fig.3.4: Total density of populations from sites1 and 2.

***Case* 2:** In this case, all the data are as same as in the case 1 except that there is an immigration from site 1 to site 2. Therefore, considering:

- $\mu_k = 0.03 \quad k = 1,2$
- $h^1 = 0$ (no fishing in side 1)
- $h^2 = 0.5$ (high harvesting in side 2)
- $m_{21} \neq 0, m_{12} = 0$ (emigration from side 1 and 2)



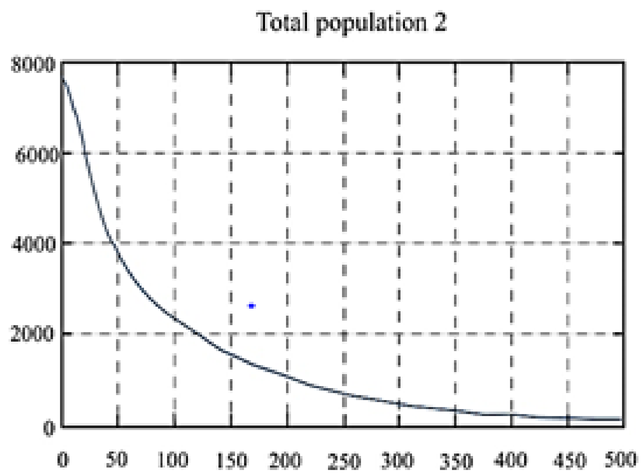


Fig.3.5: Total populations from site1 and 2. The population in site 2 is decreasing promptly than the site 1 due to the high fishing rate (left).

From the Fig.3.7 – 3.10, it is observed that the total populations of site 1 is decreasing because fishes are migrated from site 1 to site 2. For the same reason, the total population of site 2 is increasing and the new born population of the site 2 is growing up rather than case 1.

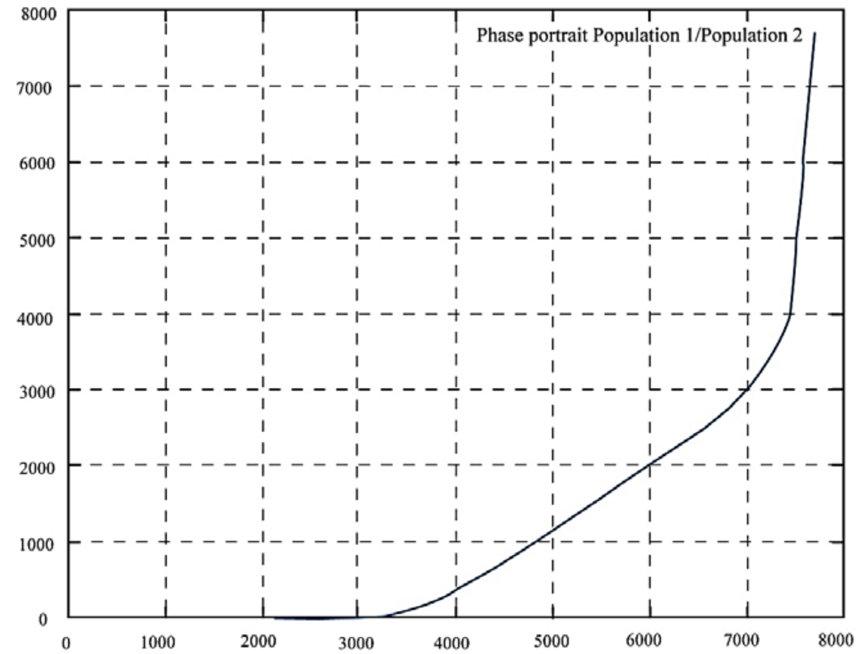


Fig.3.6: Phase portrait of population for sites 1 and 2.

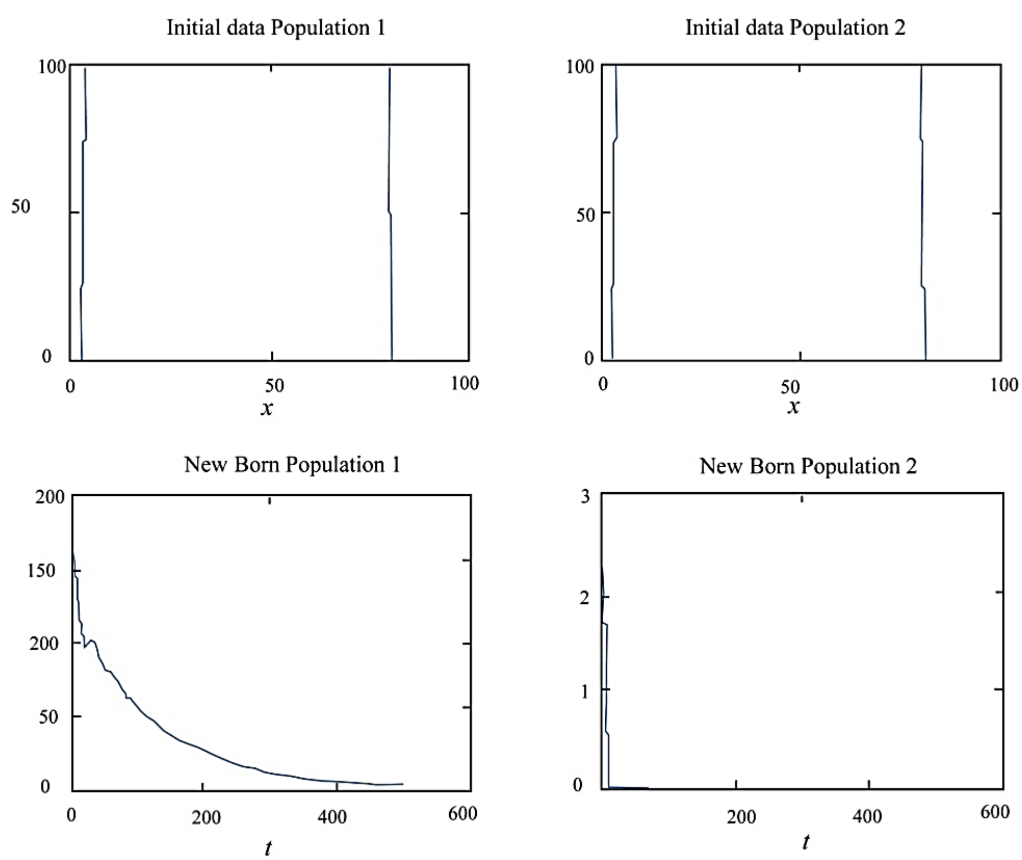


Fig.3.7: Initial populations and newborn rate.

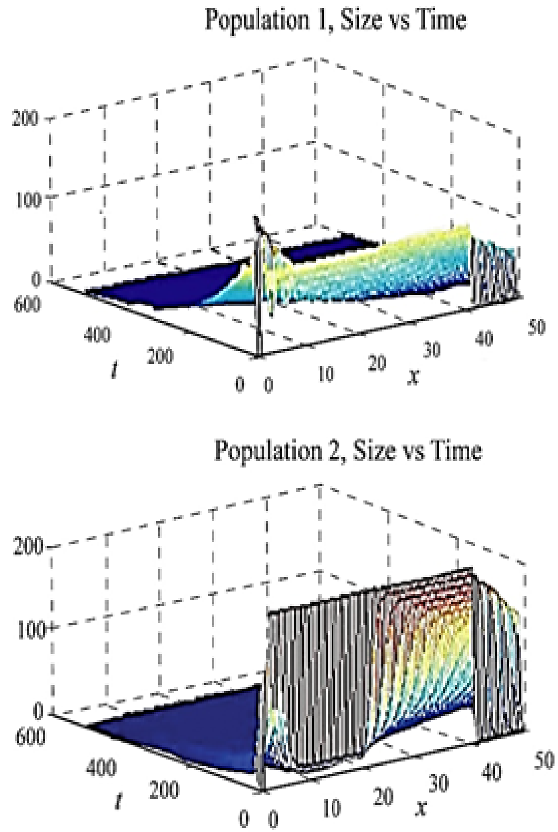


Fig.3.8: The total density of fish populations in both sites.

***Case* 3:** In this case, both the zones 1 and 2 are considered as fishing regions. There are mortality rate and movement rate are *present* in both of the zones. Therefore,

- $\mu_k = 0.03,\ k = 1,\ 2$
- $h^1 = 0.01$ (low fishing in side 1) and $h^2 = 0.5$ (high harvesting in side 2)
- $m_{21} \neq 0, m_{12} \neq 0$

Using the data, it is concluded that the total population are growing up in both sites. Here the born population are changing their sites due to mortality rate and considering the migration and emigration within both sites, which are shown in Figs.3.11– 3.12.

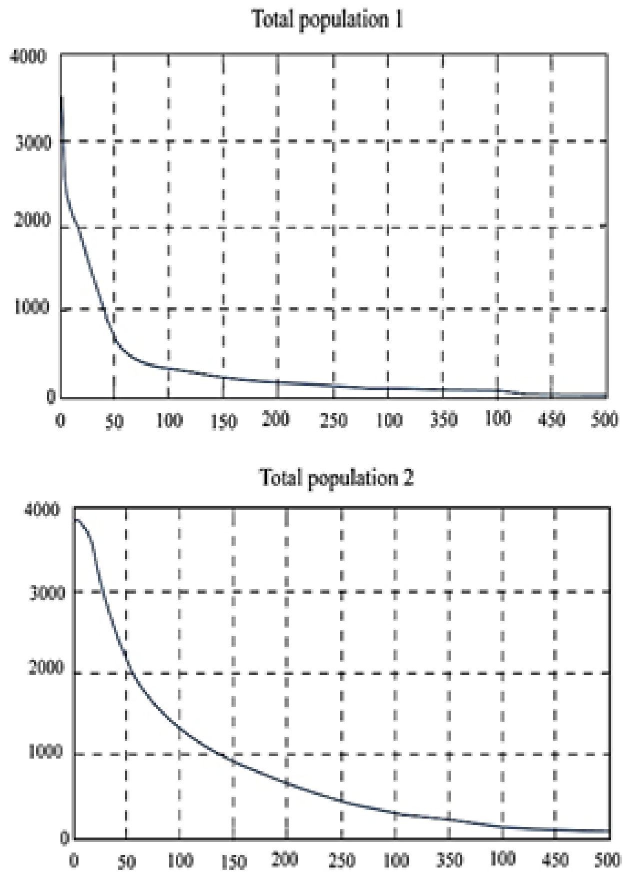


Fig.3.9: Total populations from site1 and 2.

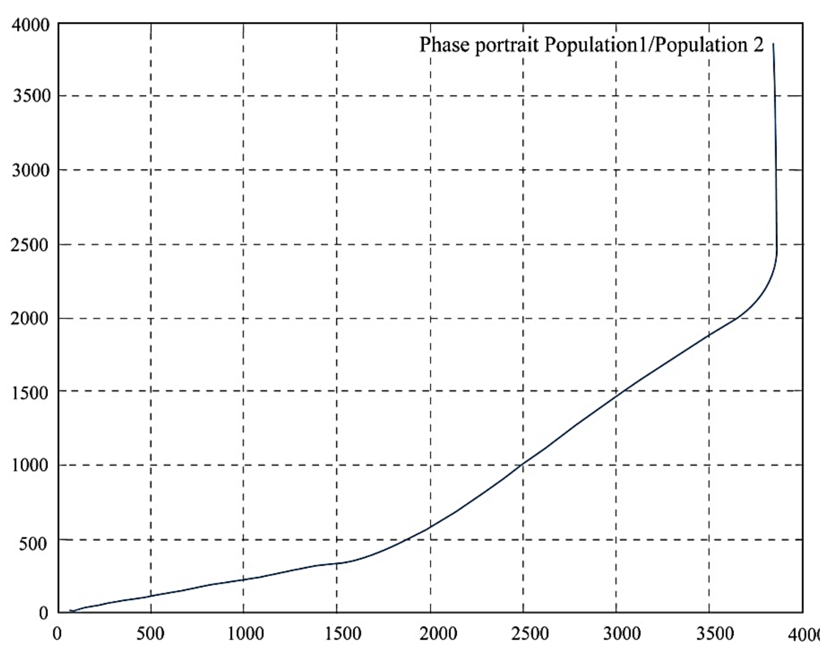


Fig.3.10: Phase portrait of population for sites 1 and 2.

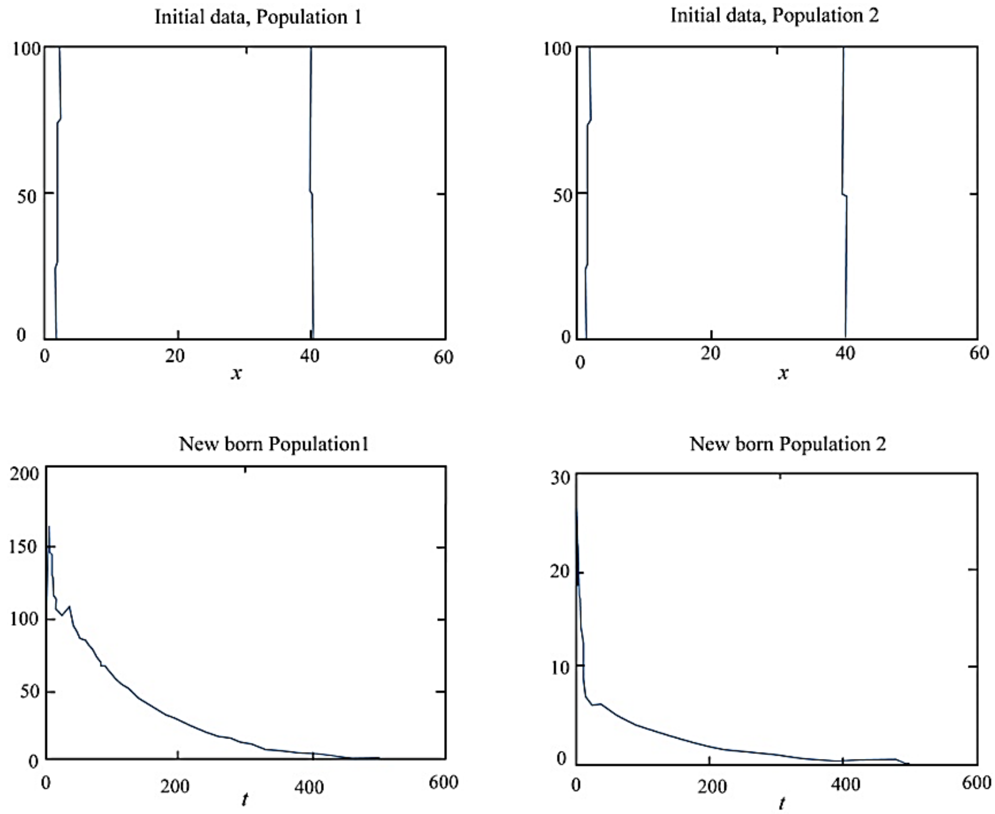


Fig.3.11: Initial populations and newborn rate.

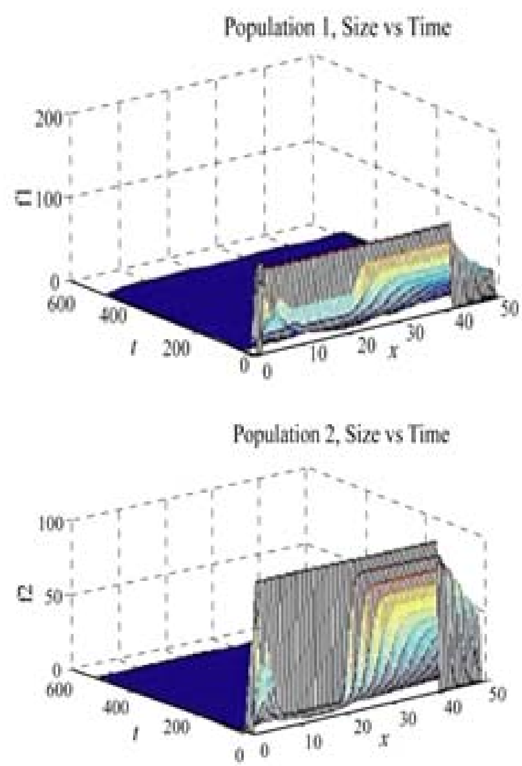


Fig.3.12: The total density of fish populations in both sites.

3.5.2 The model is nonlinear

To have an idea on asymptotic behavior of the total population in each site, the following suppositions are proposed.

$$\beta^k(x,t) = \beta^k\left(P^k(t)\right) \text{ and } \mu^k(x,t) = \mu^k\left(P^k(t)\right) \text{ for } k = 1,2.$$

$$\begin{aligned}
\frac{dP^k(t)}{dt} &= \int_0^L \left(\beta^k\left(P^k(t)\right) - \mu^k\left(P^k(t)\right) - h^k\left(P^k(t)\right) m_{k\to l}\right) p^k dx + \int_0^L m_{l\to k} p^k dx \\
&= \left(\beta^k\left(P^k(t)\right) - \mu^k\left(P^k(t)\right) - h^k\left(P^k(t)\right) m_{k\to l}\right) \int_0^L p^k dx + m_{l\to k} \int_0^L p^k dx \\
&= (\beta^k - \alpha P^k)P^k - H^k \qquad\qquad t \in (0,T)
\end{aligned}$$

(3.23)

where P^k is the total population, H^k represents harvesting rate plus movement rate and $\mu(P^k) = \alpha P^k$.

***Case* 1:** If H^k is null, i.e., there is no fishing rate and no movement rate, then the Eq.(3.23) becomes

$$\begin{aligned}
\frac{dP^k(t)}{dt} &= (\beta^k - \alpha P^k)P^k, \quad k = 1,2 \\
&= \beta^k \left(1 - \frac{P^k}{\frac{\beta^k}{\alpha}} \right) P^k == \beta^k \left(1 - \frac{P^k}{K} \right) P^k
\end{aligned} \qquad (3.24)$$

where K is the carrying capacity of site k = 1, 2 and $\lim_{t\to\infty} P^k = K$.

***Case* 2:** If $H^k \neq 0$, , i.e., there is fishing rate and movement rate, then the Eq.(3.23) becomes

$$\begin{aligned}
\int \beta^k(P^k) p^k dx &= \beta^k(P^k) \int_{x_1}^{x_2} p^k dx \quad k = 1,2 \\
&= \beta^k(P^k) P^k
\end{aligned}$$

where P^k is the total population in (x_1, x_2).

The results of the two cases where $H^k = 0$ and $H^k \neq 0$ are examined under two experimental cases.

Experiment 1: Site 1 is a protected area where there is no harvesting rate and there is an emigration from site 1 to site 2. Since site 2 is a fishing area, so its harvesting rate is high, and the results of this experiment are depicted in Figs.3.13–3.16. Figures show that during recruitment period, in site 1, total population and new born population are growing up because there is no harvesting rate and in site 2, total population and new born population are growing up because there is an emigration.

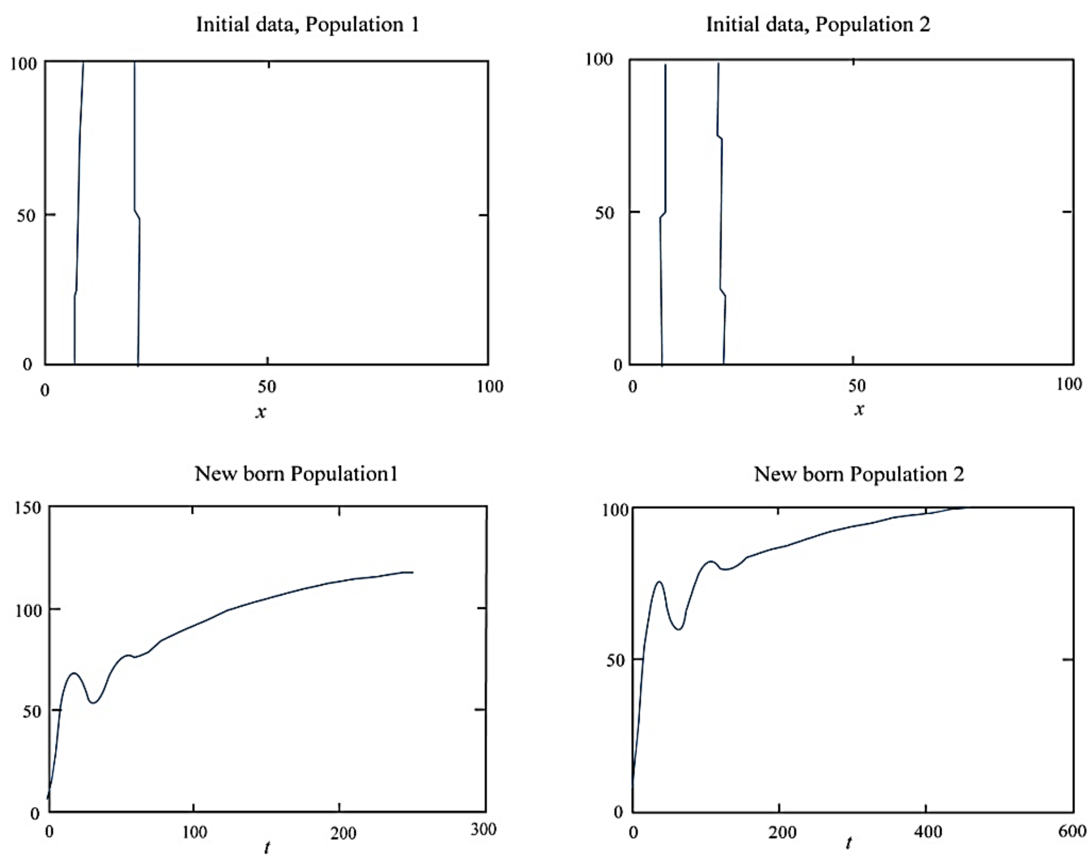


Fig.3.13: Initial populations and newborn rate.

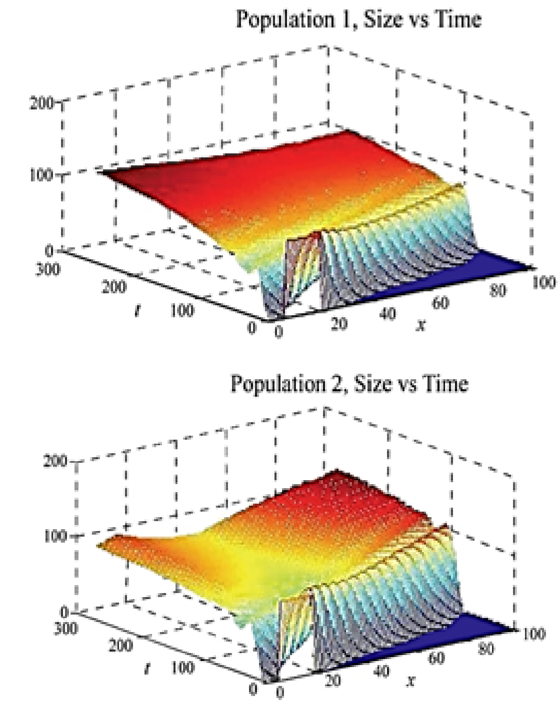


Fig.3.14: Total density of populations from site 1 and 2.

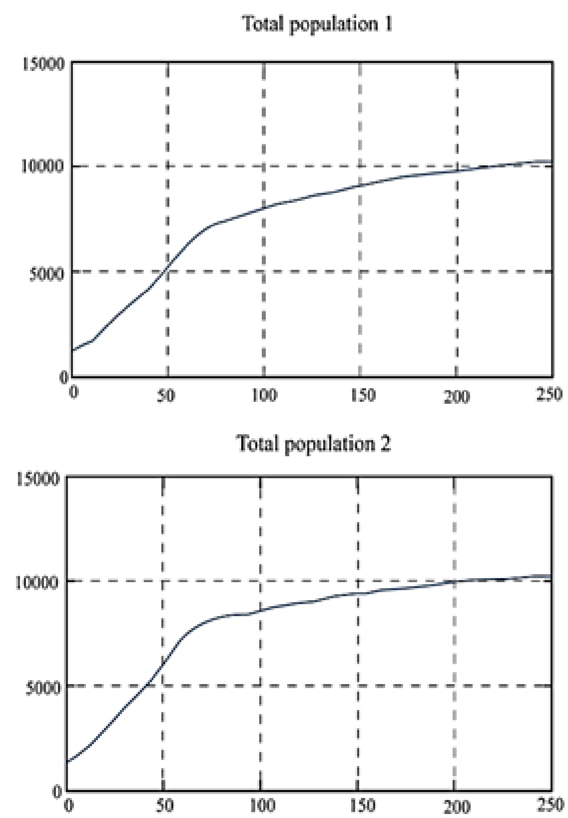


Fig.3.15: Total populations from site1 and 2.

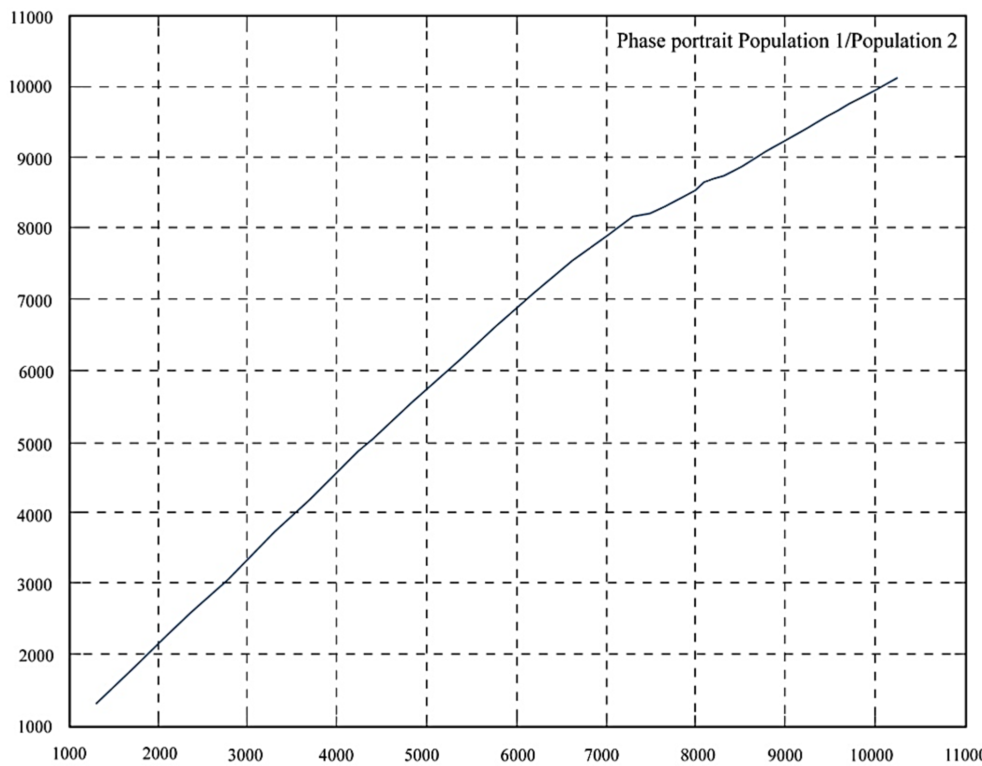
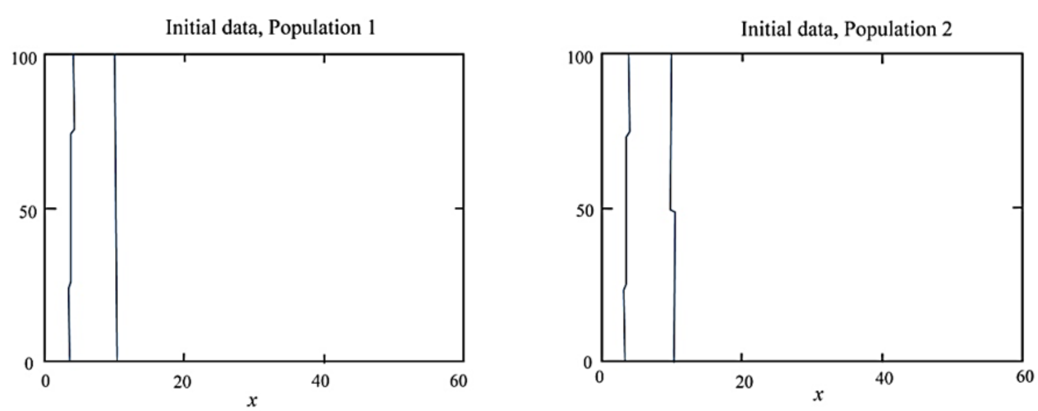


Fig.3.16: Phase portrait of population for sites 1 and 2.

Experiment 2: From the Figs. 3.17 –3.20 it is observed that the total population and the new born population are in equilibrium position at the end of the generations in both sites.



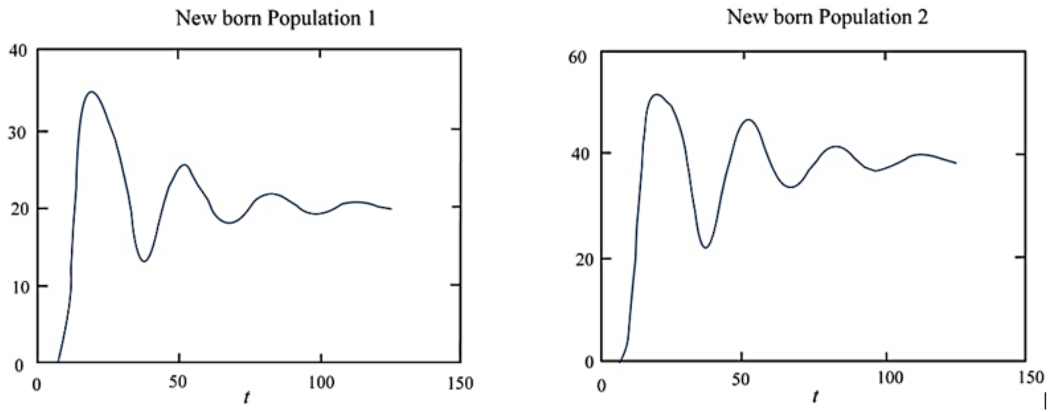


Fig.3.17: Initial populations and newborn rate.

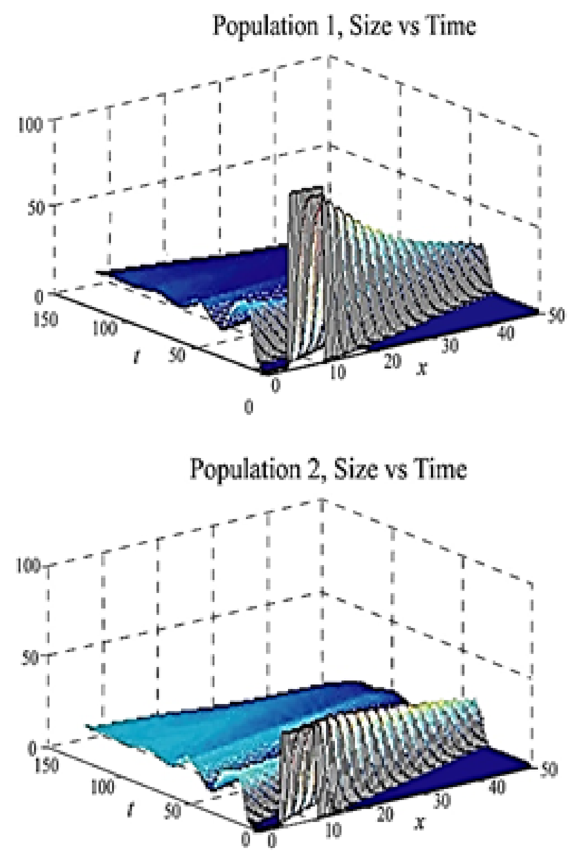


Fig.3.18: Total density of populations from sites1 and 2.

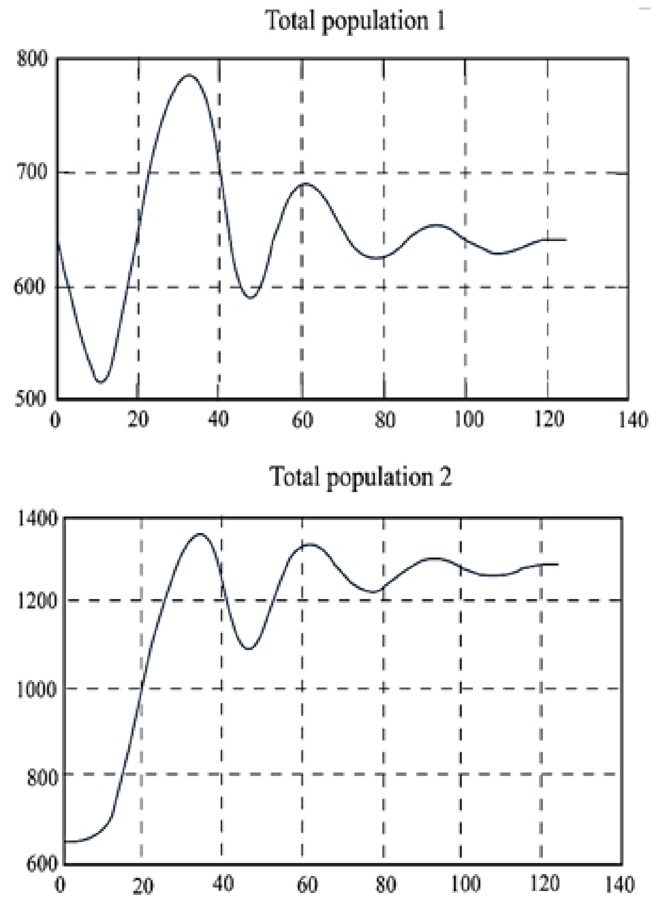


Fig.3.19: Total populations from site 1 and 2.

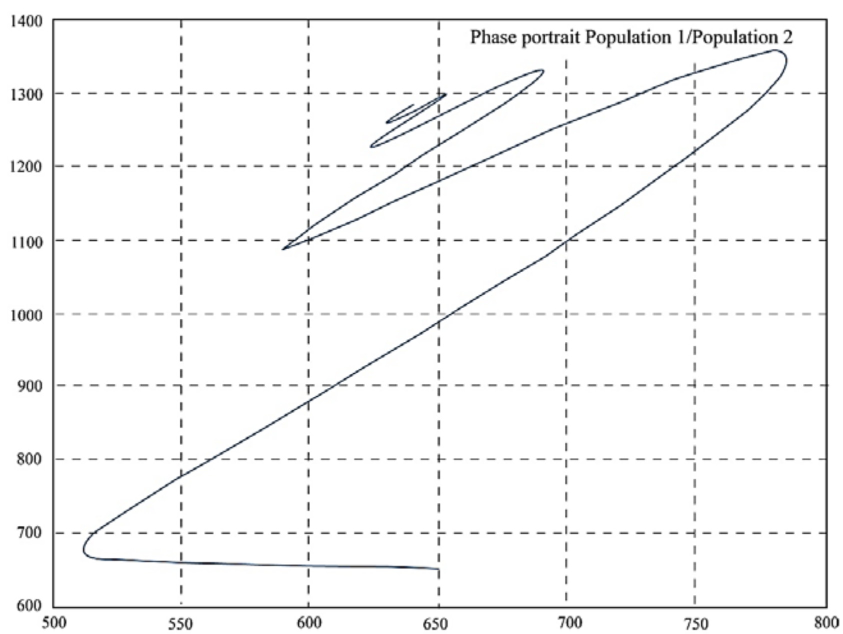


Fig.3.20: Phase portrait of population for sites 1 and 2.

3.6 CONCLUSIONS

In this chapter, a multi-region linear and nonlinear size structured fish population model has been developed and investigated under various conditions. The model is formulated in the generic way so that it can be used for different types of fish species. After that the model is derived using initial boundary-value problem and proved existence and uniqueness of the solution. Using finite volume method the continuous problem is discretized and upwind explicit scheme is determined. Consistency and stability of the scheme are also established. Using the scientific programming language FORTRAN®, numerical solutions has been found out considering the key parameters (i.e., recruitment, mortality, and movement). Finally, some experimental works have been worked out and it is realized that fish population in each zone is increasing when there is a recruitment, and the population is decreasing when subjected to high level of fishing and immigration. Further, using this model the population size calculation and its comparison with experimental results are shown in the next chapter.

Chapter 4

Mathematical estimation of production performance of Fish Population

Chapter **4**

Mathematical estimation of production performance of Fish Population

Fish population dynamics demonstrates the quantitative changes in the number of individuals in which a given fish population grows and shrinks over time as controlled by some specific factors, and is generally used in the fisheries science in order to determine sustainable yields. The aim of this chapter is to present and analyze a generic mathematical formula of a single-region size structured model which is useful for the fish production estimation. With the intention of estimation the final fish size, the von Bertalanffy's growth equation is modified and utilized based on the initial size of the fish species. For fish population calculation, initial size, birth, growth, mortality rates and the arbitrary constant of modified von Bertalanffy's growth equation are considered as input variables. The number of total fish population and the fish size at different time spans are calculated. The results coming from the mathematical calculation are compared to the experimental results of freshwater mud eel fish production. Comparing to these values, it reveals that there is no significant difference between the result obtained by implementing the proposed mathematical model and the experimental outcomes obtained. Finally, it is expected that the modified model can be used for the practical applications of fish production.

(N.B.: For clear understanding and to make this chapter independent some relevant informations are kept repeating)

4.1 INTRODUCTION

Bangladesh is a densely populated country, currently with a population of around 160 million people (Worldometer, 2016). This country is an agro-based developing country being endowed with natural fisheries resources. It is fortunate in having an extensive water resource in the form of ponds, natural depressions (i.e., haors and beels), lakes, canals, rivers and estuaries. The consumption of fish food is increasing as the population is increasing geometrically whereas the land and water are decreasing at the same rate. Therefore, to meet the growing demand, it is necessary to establish sustainable fisheries, or to increase the number of artificial fisheries which is practically impossible due to the consumption of lands by overpopulation. At present the fisheries sector in Bangladesh plays a significant role for fulfilling the demand of protein, nutrition, employment, poverty alleviation of al large number of unemployed population and foreign exchange earnings. Aquaculture produces about 3.46 million tons of fish, of which about 2 million tons were farmed in the 2013-14 (Mahmud, 2014). Bangladesh ranks third among the world's largest inland fish producing countries after China and India. Around three quarters of rural households practice some form of freshwater aquaculture covering some 10 million ponds and most of which measure less than 400 m^2 (Ghose, 2014).

Fisheries in Bangladesh are diverse, there are about 795 native species of fish and shrimp in the fresh and marine waters of Bangladesh, and 12 exotic species that have been introduced (Karim, 2003). To meet the present demand and considering future potentials, a large number of fisheries have been established in different parts of the country. Although there seems a huge success in producing large quantity of fish population, the major tasks for fisheries management is to regulate their fishery in such a way as to obtain the maximum benefit from it. However, the fishery is a complex system and it is not easy to interpret the wide range of data that can be obtained about such diverse features as growth rates, harvesting with respect to fishing gear, mortality, immigration and emigration, etc., nor is it easy to predict the effect on the fisheries management. Hence, several

mathematical models and techniques are existing for the study of fish population in order to get an idea how much fish be produced by using these parameters.
Fish population dynamics describes the quantitative changes in the number of individuals in a population or the vital rates of a population as a result of various different processes i.e., growth, natural mortality, mortality due to fishing, emigration or immigration, sexual differentiation over time. A number of authors worked on the fish population dynamics based on modelling the growth processes. Dubey et al. (2002) presented a dynamic model for a single-species fishery with optimal harvesting policy that depends partially on a logistically growing resource. Later on, the same authors proposed a non-linear mathematical model to study the fisheries resource system dynamics using the Pantryagin's maximum principle (Dubey et al., 2003). Faugeras and Maury (2005) established an advection-diffusion size-structured fish population dynamics model to simulate the skipjack tuna population in the Indian Ocean. The proposed model is fully spatialized, and the movements are parameterized with oceanographical and biological data.

Kar (2006) proposed a nonlinear mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists of two zones; a free fishing zone and a reserve zone where fishing is strictly prohibited. He observed that in the absence of any predator, even under continuous harvesting, fish population may be maintained at an appropriate equilibrium level. Later on, Kooten et al. (2010) proposed a mathematical model to study the relation between hatching size and response to harvesting mortality. The result shows that the hatching size determines dynamics through its effect on the relative strength of cannibalistic mortality. However, most of the models are of weakly coupled hyperbolic partial differential equations with non-local boundary conditions (Aylaj and Noussair, 2010). These models are setup to preserve the fish species disappearing or to provide assessment of the fish abundance and fishery exploitation in order to determine the sustainable yield of fish population (Wentworth et al., 2011). A recent article has been published on the fish population dynamics in which both size and time are taken as structure variables to account for growth, mortality, movements of fish, environmental variability and variable distribution of fishing effort within the multi-regions. This model is established considering a system of hyperbolic partial differential equations where both linear and nonlinearities boundary conditions are carefully discussed (Ahamad et al., 2015).

The dynamic model approach, therefore, is a widely applied technique to a number of environmental management issues particularly to the fisheries management problems (Bendor et al., 2009; Martinet et al., 2010; De Lara et al., 2011). To devise these models, a considerable amount of work of recording data is required, extending over a prolonged period of time in order to obtain reliable results. Thus, the data collected for the fisheries management to make it possible, in particular, to establish growth and mortality rates. A considerable number of researchers works on the von Bertalanffy's growth equation in order to estimate the both growth and reproduction of fish population (Cloern and Nichols, 1978; Essington et al., 2001; Lester et al., 2004; Taylor et al., 2005; Cailliet et al., 2006). Hart and Chute (2009) introduced a mathematical formulas for estimating von Bertalanffy growth parameters from growth increment data using a linear mixed-effects model that lack explicit age information. Although this approach produces unbiased estimates, it is sometimes difficult to implement and compute the growth and size of fish population in different time spans.

From the aforementioned literatures it is found that the fish population dynamics plays an important role in the fishing gear considering the various controlled parameters i.e., growth, natural mortality, mortality due to fishing, emigration or immigration, harvesting time, setting catch quotas, restricting the legal size of fishing, etc. In this chapter, a generic mathematical formula of a single-region size structured model which is useful for the different fish species has been used and analyzed for the fish production. In order to estimate the final fish size, the von Bertalanffy's growth equation has been modified and utilized based on the initial size of the fish species. During the fish population calculation, birth, initial size, growth, mortality rates and the constant 'K'of modified von Bertalanffy's growth equation are considered as input variables within the single-region. The outcomes of this articles are calculated in order to find out the total population and the fish size in different time horizon. At the end of the chapter, the results coming from the mathematical equation are compared to the experimental results of fish population production performance.

4.2 EXPERIMENTAL

In this chapter, a hypothetical mathematical model and data coming from the field of the fish population are compared considering the linear model approach. Therefore, among the parameters, only birth, growth and mortality rates are considered as input factors and the total population, size on time are considered as output factors. On compare, the results are summarized.

4.2.1 Mathematical model used

Refereeing to the work of Ahamad et al. (2015), let P be the total fish-population in the domain, Ω. The number of individuals of length between 0 and L, in the domain Ω_k at time t is given by

$$P^k(t) = \int_0^L p^k(x,t)dx \tag{4.1}$$

where $p^k(x, t)$ is the density of fish population of length x at time t in the zero Ω_k. The basic model of population dynamic of fisheries is

$$\begin{aligned} P^k(t+\delta t) = P^k(t) &+ \text{birth rate} + \text{migration rate} \\ &- \text{mortality rate} - \text{emigration rate} \end{aligned} \tag{4.2}$$

where $P_k(t + \delta t)$ is the number of fish at time $t + \delta t$, δt is the time variation. The mortality rate includes both the fishing mortality and natural mortality. The mathematical equation is

$$\begin{cases} \dfrac{\partial p^k(x,t)}{\partial t} + \dfrac{\partial\left(v^k(x,t)p^k(x,t)\right)}{\partial x} = -g^k(x,t)p^k(x,t) + (Mp)^k & (x,t)\in \mathfrak{O} \\ p^k(x,0) = p_0^k \quad x\in(0,L) \\ v^k(0,t)p^k(x,0) = \int_0^L \beta(x,t)p^k(x,t)dx \quad t\in(0,T) \end{cases} \tag{4.3}$$

where,

$$\left(Mp\right)^k = \begin{pmatrix} -\sum_{i\neq 1}^{nk} m_{1\to i} & m_{2\to 1} & m_{3\to 1} & \cdots & m_{nk\to 1} \\ m_{1\to 2} & -\sum_{i\neq 2}^{nk} m_{2\to i} & m_{3\to 2} & \cdots & m_{nk\to 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1\to nk} & m_{2\to nk} & m_{3\to nk} & \cdots & -\sum_{i\neq k}^{nk} m_{k\to l} \end{pmatrix} \begin{pmatrix} p^1 \\ p^2 \\ \vdots \\ p^{nk} \end{pmatrix}$$

4.3 MODIFICATION OF VON BERTALANFFY'S FISH SIZE EQUATION

One of the most widely used models for quantifying growth in fish is the von Bertalanffy's growth curve which is most widely used models for quantifying growths and is especially important in fisheries studies. He derived this equation from simple physiological arguments. It assumes that the growth rate of an organism declines with size so that the rate of change in length, *l*, may be described by:

$$\frac{dl}{dt} = K(L_\infty - l) \tag{4.4}$$

where t is time, l is the length (or some other measure of size), K is the growth rate and L_∞ is the final fish size. After simplification, the equation reduces to:

$$l_t = L_\infty(1 - e^{-K(t-t_0)}) \tag{4.5}$$

The parameter t_0 is included to adjust the equation for the initial size of the fish and is defined as age at which the fish population would have had zero size. Thus, in order to fit into the model developed by von Bertalanffy it is required to estimate the three parameters such as L_∞, the final fish size, K the growth rate, and t_0, the initial fish size. Therefore, it is observed that plotted curve starts from the final size and moves towards the initial size of the organism, which resembles a concave downward curve (i.e., exponential decay) in nature. Moreover, it is must to know the final size of the organism in order to estimate the size of species in time periods. The theory behind various growth models is reviewed, however, this experimental research is carried out on the basis of an idea that the curve shows an inverse relationship to the von Bertalanffy's model. Therefore, the von Bertalanffy's growth equation is modified which reduced to:

$$L(\mathrm{t}_{n+1}) = L(\mathrm{t}_n) + L(\mathrm{t}_n) \times [1 - e^{-K(\mathrm{t}-\mathrm{t}_0)}], \quad n = 0, 1, 2 \cdots \quad (4.6)$$

where $L(\mathrm{t}_0)$ is the initial fish size and 'K' is said to be arbitrary constant which is positive. The value of 'K' is to be estimated as the similar way to estimate the values of mortality, growth, birth rate etc. from experimental sample environment.

The modified equation bears the advantages that the resulted value starts from an initial size of the fish species. Besides, it is not necessary to know the final size of the fish at different time periods. The proposed model starts from an initial size and will continue exponentially which resembles as an exponential upward curve. Moreover, it is possible to estimate the fish size in different time span.

4.4 SAMPLE COLLECTION AND PRESENTATION

The experiment was conducted in three tanks in the Fish Breeding House under the Department of Genetic Engineering and Biotechnology, Shahjalal University of Science and Technology (SUST), Sylhet, Bangladesh. In addition, two tanks and one backyard tank were selected in a rural house, Chatak, Sunamgonj. Experiment with a plastic tank was designed in a pond of tilapia hatchery at Kamal Bazar, Sylhet. Collected fish species were divided into three categories and placed in different places such as two house tanks and one backyard tank. To compare the significance of the findings, paired sample statistics for mean and standard deviation, paired sample correlations and paired sample significance test were calculated for the growth of experimental fish considering length (size). After the collection of primary data from different environments, the Table – 4.1, is prepared (Miah et al., 2015).

Refereeing to the article (Ahamad et al., 2015), it was said that the considered domain was a multi-region, domain Ω_k, and the model developed was valid for both linear and nonlinear cases. However, the article (Miah et al., 2015) revealed the data only for a single environment (single domain, Ω_k, $k = 1$). Hence the present work has been followed a single domain, which implies the model is linear case only.

Table – 4.1: Stocking rates of fish population in different Environments.

Culture or Environments	F*l*S (cm)	FS (cm)	TN	SF	GR	MR	RP
Tank 1	15	30.6	40	35	0.7070	0.125	6
Tank 2	15.5	25.05	40	30	0.4215	0.25	6
Backyard tank	15.7	25.4	40	31	0.4315	0.225	6
Earthen ditch	14.5	32.75	40	37	0.8616	0.075	6

*F*l*S = Fingerline size, FS = Final size, TN = Total number of fish, SF = Survival fish, GR = Growth rate, MR = Mortality rate, RP = Rearing period (month)

4.5 RESULT AND DISCUSSION

For this chapter only one site is under the influence of fishery. The case is one sided, therefore, there is no harvesting, and no migration and emigration takes place in the system. Referring to the Table – 4.1, the side only uses the initial size, final size, growth rate, and mortality rate. On the application of Eq.(4.3), the value of birth rate, β must be considered as small as possible but not zero.

***Tank 1*:** For this case the value of β is 0.000213. Taking the values of growth and mortality rate form Table – 4.1 and putting these values in Eq.(4.3), it is found that after 6 months or 0.5 year, the total population is 34.79, shown in Fig.4.1, which is very close to the value found in the published article (Miah et al., 2015).

Since they (Miah et al., 2015) have used the initial fish size or newborn fingerlings fish started on the time being, i.e. $t_0 = -0.2$ year for each cases. Following the similar way, taking the initial fish length is 15 cm and putting the value of K= 0.305 per year, using the Eq.(4.6), it is found that after 6 months, the size of fish population is 30.5956 cm, shown in Fig.4.2, which is very close to the fish size found in the same article.

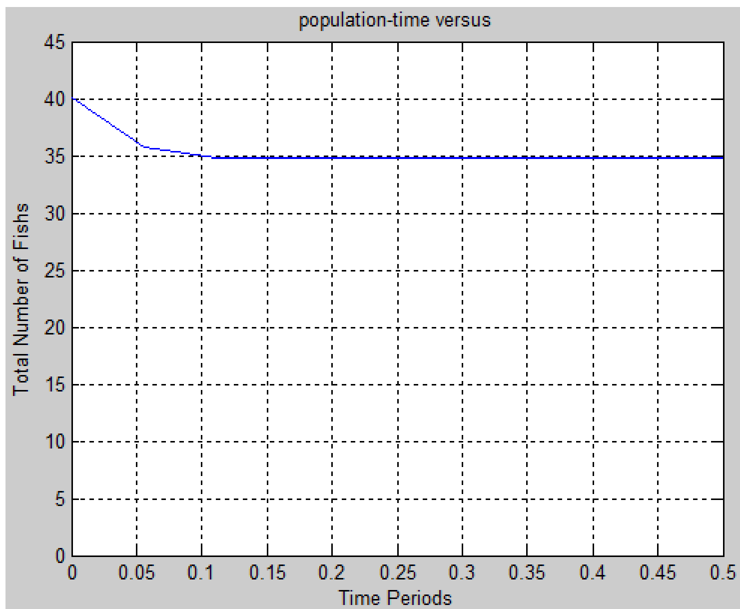


Fig.4.1: Total fish population vs time periods for Tank 1.

***Tank 2*:** For this case the value of β is 0.0002. Taking the values of growth and mortality rate form Table 1 and putting these values in Eq.(4.3), it is found that after 6 months or 0.5 year, the total population is 30.02, shown in Fig.4.3, which is very close to the value 30, found in the article published (Miah et al., 2015).

Following the similar approach as stated for Tank – 4.1, taking the initial fish length is 15.5 cm and putting the value of K= 0.196 per year, using the Eq.(4.6), it is found that after 6 months, the size of fish population is 25.075 cm, shown in Fig.4.4.

Comparing to the reference article (Miah et al., 2015), it is observed that the final fish sizes are approximately same.

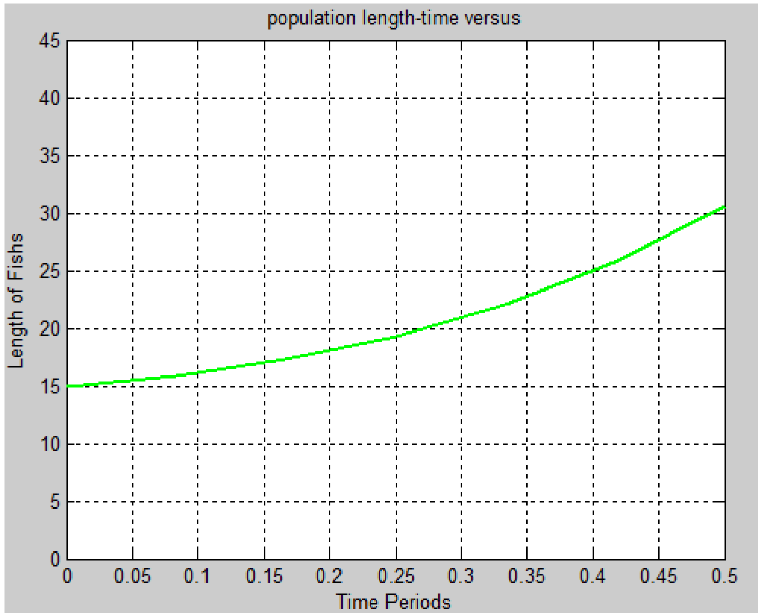


Fig.4.2: The final fish size vs time periods for Tank 1.

Backyard Tank**:** In this case the value of β is 0.00019. Taking the values of growth and mortality rate form Table – 4.1 and putting these values in Eq.(4.3), it is found that after 6 months or 0.5 year, the total population is 31.075, shown in Fig.4.5, which is very close to the value 31 found in the article (Miah et al., 2015).

Following the similar approach as stated for **Tank** 1 and 2, taking the initial fish length is 15.7 cm and putting the value of K= 0.196 per year, using the Eq.(4.6), it is found that after 6 months, the size of fish population is 25.3988 cm, shown in Fig.4.6. Comparing to the reference article (Miah et al., 2015), it is observed that the final fish sizes are approximately same.

Earthen ditch: Similarly, taking the value of β is 0.000265 and the values of growth and mortality rate form Table – 4.1 and putting these values in Eq.(4.3), after similar time period, the total population is 37.09, is found as shown in Fig.4.7, which is very close to the value 31 found in the article (Miah et al., 2015).

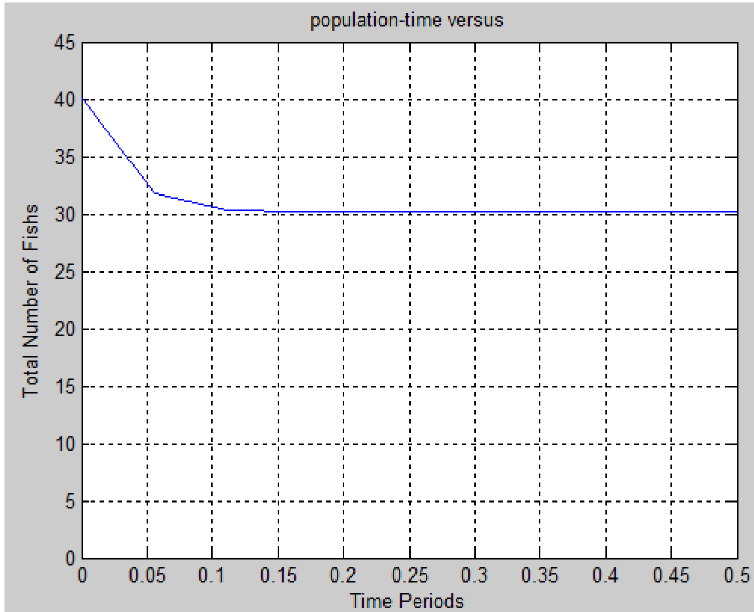


Fig.4.3: Total fish population vs time periods for Tank 2.

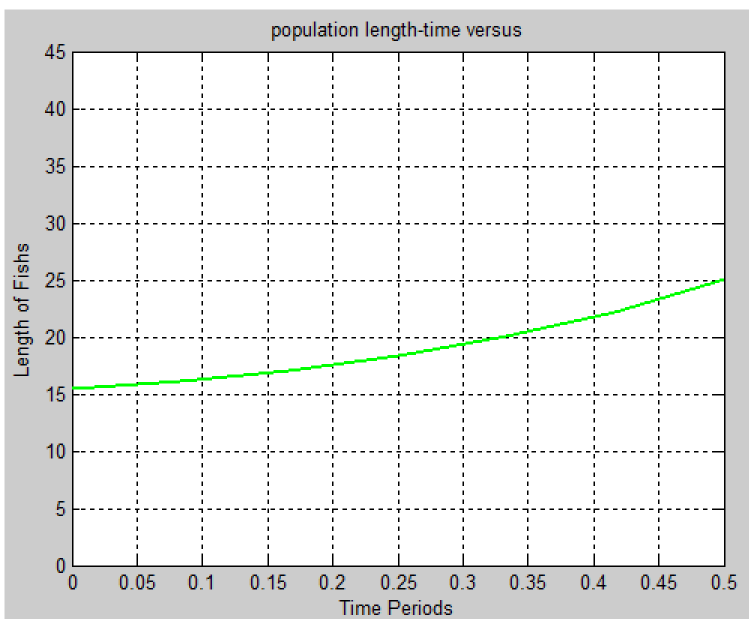


Fig.4.4: The final fish size vs time periods for Tank 2.

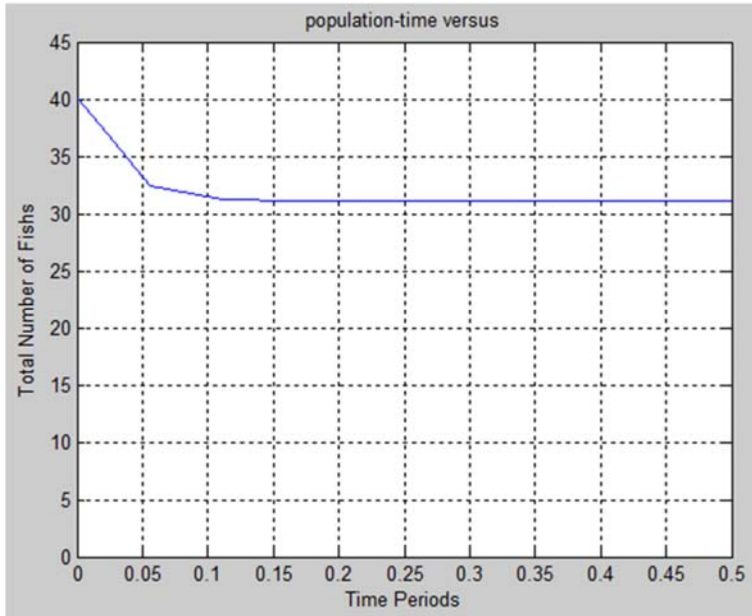


Fig.4.5: Total fish population vs time periods for Backyard Tank.

Following the similar approach, taking the initial fish length is 15.7 cm and putting the value of *K*= 0.53 per year, using the Eq.(4.6), the size of fish population is 32.6795cm, is found as shown in Fig.4.8. Comparing to the reference article (Miah et al., 2015), it is observed that the final fish sizes are approximately same.

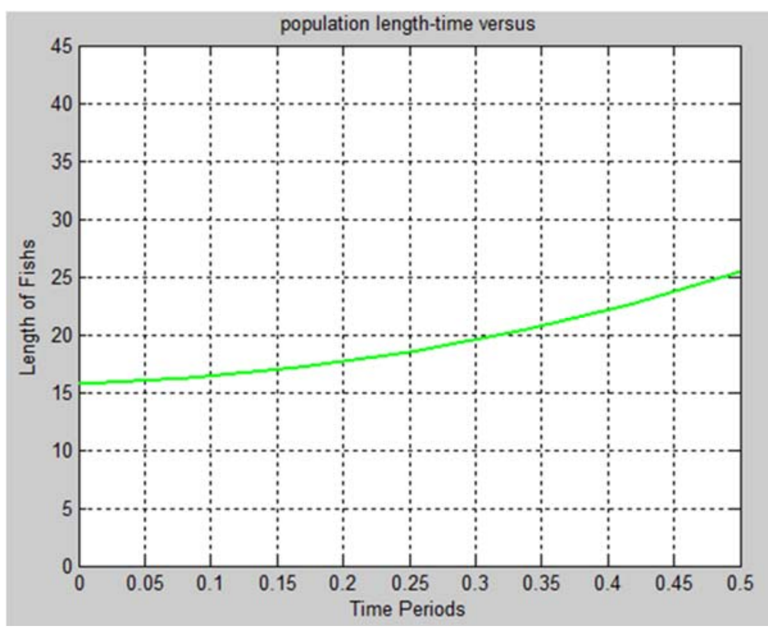


Fig. 4.6: The final fish size vs time periods for Backyard Tank.

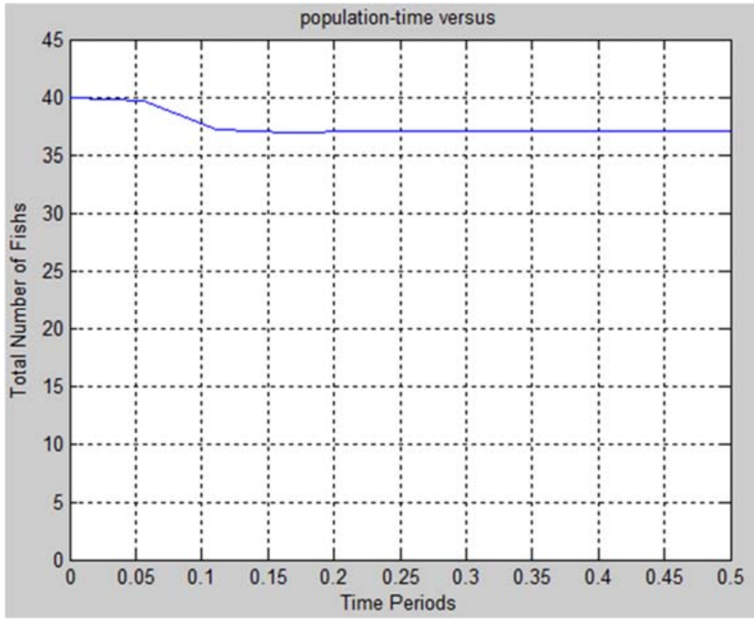


Fig.4.7: Total fish population vs time periods for Earthen ditch.

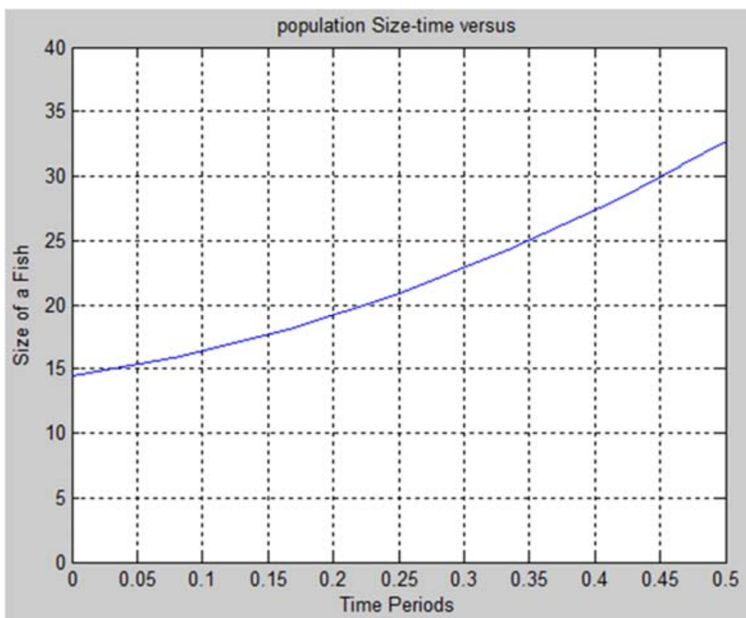


Fig.4.8: The final fish size vs time periods for Earthen ditch.

The comparison between the values computed by mathematical modelling considering a single region, and the result computed for the production performance of freshwater mud eel, *Monopterus Cuchia*, are given in Table – 4.2. Comparing to the values, i.e., total fish population and the final fish size obtained by mathematical modelling and experimental data reveals that the model developed is justified for practical applications, and the model can be used for fish population production sector.

Table – 4.2: Comparison between the computed value and the values form reference article (Miah et al., 2015).

Culture or Environments	Fingerline size (cm)	Reference Article		Computed value	
		TNFS	Final size (cm)	Total fish Population	Final size (cm)
Tank 1	15.0	35	30.6	34.79	30.5956
Tank 2	15.5	30	25.05	30.02	25.075
Backyard Tank	15.7	31	25.4	31.07	25.3988
Earthen ditch	14.5	37	32.75	37.09	32.6795

*TNFS = Total number of fish survived.

4.6 CONCLUSIONS

In this chapter, a single-region linear size structured fish population model has been developed and investigated under various environments. The model is formulated in the generic way so that it can be used for different types of fish species. Furthermore, the final fish size model developed by von Bertalanffy is modified based on the initial size of the fish species. The proposed model starts from an initial size and continue exponentially upward direction, which is considered to estimate the fish size in different time span.

A MATLAB® software is used to calculate the values of total population and the final size of fish population. The logical sequences of the program related to this work is given in Appendix I. The values obtained by mathematical modelling are compared to the experimental data for the production performance of freshwater mud eel.

Therefore, the fisheries management can predict how much yield will be after a certain period of time. In the same time the management can estimate the future value of fish population, as well as the profit gain from the production. Finally, it is expected that the modified model can be used for the practical applications, and will be able to estimate the fish production.

Chapter 5

Various Element Matrices Encountered in Finite Element Formulation of Engineering Problems

Chapter 5

Various Element Matrices Encountered in Finite Element Formulation of Engineering Problems

This chapter is mainly concerned to identify various type of element matrices needed to finite element solution procedure of engineering problems. To do so, it reviews the governing equations and finite element formulations of two problems namely Shear deformation plate theory (SDT) and Magneto-hydrodynamics double-diffusive mixed convection for unsteady flow. Finally, it presents general forms of integrals to form all the element matrices.

5.1 SHEAR DEFORMATION PLATE THEORY (SDT)

In the SDT, the normality assumption of Classical Plate Theory is relaxed, i.e., transverse normal may rotate without remaining normal to the midplane (see Fig.5.1) is based on the displacement field

$$\begin{aligned} u_1(x,y,z,t) &= u_x(x,y,t) + z\phi_x(x,y,t) \\ u_2(x,y,z,t) &= u_y(x,y,t) + z\phi_y(x,y,t) \\ u_3(x,y,z,t) &= u_z(x,y,t) \equiv w(x,y,z) \end{aligned} \tag{5.1}$$

where $(u_x, u_y, u_z = w)$ are the displacements of a point on the midplane in the (x, y, z) coordinate directions, and ϕ_x and ϕ_y are rotations of the transverse normal about the y and x axes, respectively. Since (u_x, u_y) are uncoupled from (w, ϕ_x, ϕ_y) we develop the equations governing (w, ϕ_x, ϕ_y).

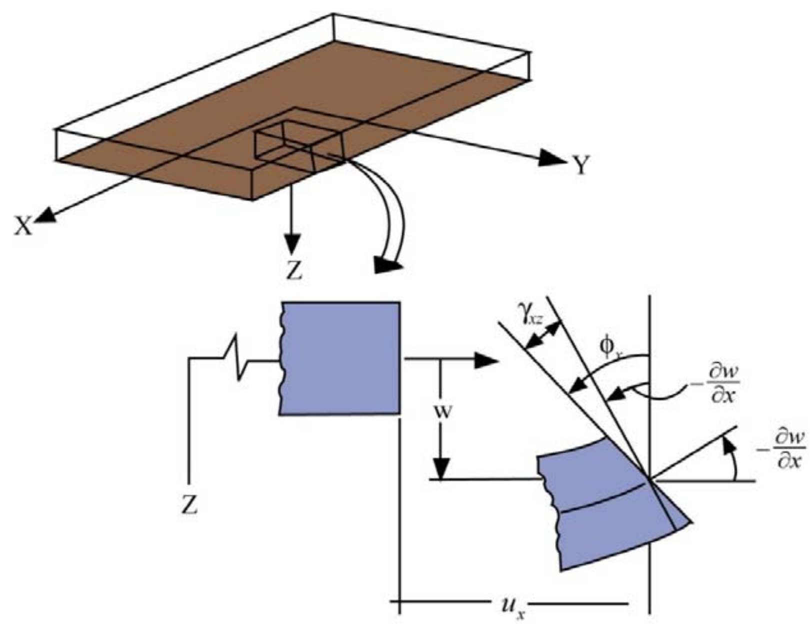


Fig.5.1: Underformed and deformed geometries an edge in SDT.

The bending strains associated with Eq.(5.1) are

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} z\dfrac{\partial \phi_x}{\partial x} \\ z\dfrac{\partial \phi_y}{\partial y} \\ z\left(\dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x}\right) \\ \phi_x + \dfrac{\partial w}{\partial x} \\ \phi_y + \dfrac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ 2\delta\varepsilon_{xy} \\ 2\delta\varepsilon_{xz} \\ 2\delta\varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} z\dfrac{\partial \delta\phi_x}{\partial x} \\ z\dfrac{\partial \delta\phi_y}{\partial y} \\ z\left(\dfrac{\partial \delta\phi_x}{\partial y} + \dfrac{\partial \delta\phi_y}{\partial x}\right) \\ \delta\phi_x + \dfrac{\partial \delta w}{\partial x} \\ \delta\phi_y + \dfrac{\partial \delta w}{\partial y} \end{Bmatrix} \qquad (5.2)$$

Note that the transverse shear strains are nonzero and $\varepsilon_{zz} = 0$.

5.1.1 Virtual Work Statement

Substituting the displacement Eq.(5.1) and strains Eq.(5.2) into the statement of the principle of virtual displacements ,we obtain

$$
\begin{aligned}
0 = \int_{V_e} & (\rho z^2 \delta\phi_x \frac{\partial^2 \phi_x}{\partial t^2} + \rho z^2 \delta\phi_y \frac{\partial^2 \phi_y}{\partial t^2} + \rho\,\delta w \frac{\partial^2 w}{\partial t^2} + \delta\varepsilon_{xx}\sigma_{xx} \\
& + \delta\varepsilon_{yy}\sigma_{yy} + 2\delta\varepsilon_{xy}\sigma_{xy} + 2\delta\varepsilon_{xz}\sigma_{xz} + 2\delta\varepsilon_{yz}\sigma_{yz})dV \qquad (5.3) \\
& - \int_{\Omega_e} \delta w\, q\, dx\, dy - \oint_{\Gamma_e} (\delta\phi_n M_{nn} + \delta\phi_s M_{ns} + \delta w Q_n) ds
\end{aligned}
$$

The first three terms in Eq.(5.3) represent the virtual work down by the inertial forces in the three coordinate directions, while the remaining terms in the volume integral represent the virtual strain energy stored in the plate. The last two integrals, one defined on the midplane Ω_e and the other on the boundary V_e, denote the virtual work down by the transversely distributed load q, M_{nn} and M_{ns} denote the normal and twisting moments, respectively. The moment M_{nn} on an edge with unit normal vector $\hat{\boldsymbol{n}}$ can be related to moments on edges x = constant and y = constant, Reddy (2000). Carrying out the integration with respect to z, we arrive at

$$
\begin{aligned}
0 = \int_{\Omega_e} & [I_0 \delta w \frac{\partial^2 w}{\partial t^2} + I_2 \left(\delta\phi_x \frac{\partial^2 \phi_x}{\partial t^2} + \delta\phi_y \frac{\partial^2 \phi_y}{\partial t^2} \right) + M_{xx} \frac{\partial \delta\phi_x}{\partial x} + M_{yy} \frac{\partial \delta\phi_y}{\partial y} \\
& + M_{xy} \left(\frac{\partial \delta\phi_x}{\partial y} + \frac{\partial \delta\phi_y}{\partial x} \right) + Q_x \left(\delta\phi_x + \frac{\partial \delta w}{\partial x} \right) + Q_y \left(\delta\phi_y + \frac{\partial \delta w}{\partial y} \right) \\
& - \delta w\, q] dx dy - \oint_{\Gamma_e} (\delta\phi_n M_{nn} + \delta\phi_s M_{ns} + \delta w Q_n) ds
\end{aligned}
$$

where

$$M_{nn} = M_{xx}n_x^2 + M_{yy}n_y^2 + 2M_{xy}n_xn_y,$$
$$Q_n = Q_xn_x + Q_yn_y + I_2\left(\frac{\partial^3 w}{\partial x\partial t^2}n_x + \frac{\partial^3 w}{\partial y\partial t^2}n_y\right)$$
$$M_{ns} = (M_{yy} - M_{xx})n_xn_y + M_{xy}\left(n_x^2 - n_y^2\right)$$
$$I_0 = \int_{-h/2}^{h/2}\rho dz = \rho h, \quad I_2 = \int_{-h/2}^{h/2}\rho z^2 dz = \tfrac{1}{12}\rho h^3,$$

$$Q_x = K_s\int_{-h/2}^{h/2}\sigma_{xz}dz = K_sA_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right), A_{55} = G_{23}h$$
$$Q_y = K_s\int_{-h/2}^{h/2}\sigma_{yz}dz = K_sA_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right), A_{44} = G_{13}h \tag{5.4}$$

$$M_{xx} = D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}, \quad M_{yy} = D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y},$$
$$M_{xy} = D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \tag{5.5}$$

$$Q_n = Q_xn_x + Q_yn_y, \quad \phi_n = \phi_xn_x + \phi_yn_y, \phi_s = \phi_yn_x - \phi_xn_y \tag{5.6}$$

where (n_x, n_y) are the direction cosines of the unit normal, $\hat{\boldsymbol{n}} = n_x\hat{i} + n_y\hat{j}$,on the boundary Γ_e.

Here I_0 and I_2 are the mass moments of inertia, D_{ij} are the plate material stiffness, and K_s denote the shear correction coefficient. The coefficient is introduced to account for the discrepancy between the distribution of transverse shear stresses in SDT and the actual distribution.

The virtual work statement Eq.(5.3) contains three weak forms for the three displacements (w, ϕ_x, ϕ_y). They are identified by collecting the terms involving δw, $\delta\phi_x$, and $\delta\phi_y$ separately and equating them to zero:

$$0=\int_{\Omega_e}\left(I_0\delta w\frac{\partial^2 w}{\partial t^2}+Q_x\frac{\partial \delta w}{\partial x}+Q_y\frac{\partial \delta w}{\partial y}-q\,\delta w\right)dx\,dy-\oint_{\Gamma_e}\delta w Q_n ds \tag{5.7a}$$

$$0=\int_{\Omega_e}\left(I_2\delta\phi_x\frac{\partial^2 \phi_x}{\partial t^2}+M_{xx}\frac{\partial \delta\phi_x}{\partial x}+M_{xy}\frac{\partial \delta\phi_x}{\partial y}+Q_x\delta\phi_x\right)dxdy$$
$$-\oint_{\Gamma_e}\delta\phi_x(M_{nn}n_x-M_{ns}n_y)\,ds \tag{5.7b}$$

$$0=\int_{\Omega_e}\left(I_2\delta\phi_y\frac{\partial^2 \phi_x}{\partial t^2}+M_{xy}\frac{\partial \delta\phi_y}{\partial x}+M_{yy}\frac{\partial \delta\phi_y}{\partial y}+Q_y\delta\phi_y\right)dx\,dy$$
$$-\oint_{\Gamma_e}\delta\phi_y(M_{nn}n_y+M_{ns}n_x)\,ds \tag{5.7c}$$

The governing differential equation of SDT are [obtained from the weak forms Eqs.(5.7a) – (5.7c)]

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=I_0\frac{\partial^2 w}{\partial t^2} \tag{5.8a}$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=I_2\frac{\partial^2 \phi_x}{\partial t^2} \tag{5.8b}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_y=I_2\frac{\partial^2 \phi_y}{\partial t^2} \tag{5.8c}$$

The boundary conditions for the Classical Plate Theory (CDT) are given below:

$$\begin{aligned}\text{clamped: }& w=0, \quad \phi_n=0\\ \text{simply supported: }& w=0, \quad M_{nn}=0\\ \text{free: }& Q_n=0, \quad M_{nn}=0\end{aligned} \tag{5.9}$$

The three-step procedure can be used to develop the weak forms of Eqs.(5.8a) – (5.8c), which will be equivalent to those listed in Eqs.(5.7a) – (5.7c). To see the equivalence, the following identities must be used:

$$\begin{aligned} M_{nn}n_x - M_{ns}n_y &= M_{xx}n_x + M_{xy}n_y \equiv \hat{M}_{nn} \\ M_{nn}n_y + M_{ns}n_x &= M_{xy}n_x + M_{yy}n_y \equiv \hat{M}_{ns} \\ \phi_x = \phi_n n_x - \phi_s n_y, \quad & \phi_y = \phi_n n_y + \phi_s n_x \end{aligned} \tag{5.10}$$

The vector form of the virtual work statement Eq.(5.3)[after replacing the stress resultants Q_x, Q_y, M_{xx}, and M_{yy} in terms of generalized displacements (w, ϕ_x, ϕ_y)] using Eqs.(5.4) and (5.5) is given by

$$\begin{aligned} 0 = \int_{\Omega_e} & [I_0(\delta \mathbf{w})^{\mathrm{T}}\ddot{\mathbf{w}} + I_2 \delta \Phi^{\mathrm{T}} \ddot{\Phi} + (\delta \Phi + \mathbf{D}_1 \delta \mathbf{w})^{\mathrm{T}} \mathbf{A}(\Phi + \mathbf{D}_1 \mathbf{w}) \\ & + (\mathbf{D}\delta\Phi)^{\mathrm{T}} \mathbf{C}(\mathbf{D}\Phi) - \mathbf{w}^{\mathrm{T}} q] d\mathbf{x} - \oint_{\Gamma_e} (\delta \Phi_n^{\mathrm{T}} \mathbf{M}_n + \mathbf{w}^{\mathrm{T}} Q_n) ds \end{aligned} \tag{5.11}$$

where

$$\Phi = \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix}, \Phi_n = \begin{Bmatrix} \phi_n \\ \phi_s \end{Bmatrix}, \mathbf{M}_n = \begin{Bmatrix} M_n \\ M_s \end{Bmatrix}, \mathbf{D}_1 = \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{5.12a}$$

$$\mathbf{D} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} A_{55} & 0 \\ 0 & A_{44} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \tag{5.12b}$$

5.1.2 Finite Element Model

We note from the boundary integrals in Eqs.(5.7a) – (5.7c) that the variables of the theory are (w, ϕ_x, ϕ_y) and the secondary variables are (Q_n, M_{nn}, M_{ns}) (or a linear combination of Q_x, Q_y, M_{xx}, M_{yy}, and M_{xy}). Therefore, the Lagrange interpolation of w, ϕ_x and ϕ_y is admissible for SDT.

We assume finite element interpolation of w, ϕ_x and ϕ_y in the form

$$w(x,y,t) = \sum_{j=1}^{n} w_j(t)\psi_j^1(x,y) \tag{5.13}$$

$$\phi_x(x,y,t) = \sum_{j=1}^{m} S_j^x(t)\psi_j^2(x,y),\ \phi_y(x,y,t) = \sum_{j=1}^{m} S_j^y(t)\psi_j^2(x,y)$$

or

$$w(x,y,t) = (\Psi^1)^{\mathrm{T}}\mathbf{w},\quad \Phi = \begin{Bmatrix}\phi_x \\ \phi_y\end{Bmatrix} = \Psi^2\mathbf{S} \tag{5.14}$$

where

$$(\Psi^1)^{\mathrm{T}} = \{\psi_1^1, \psi_2^1, \psi_3^1, \cdots, \psi_n^1\},\quad \Psi^2 = \begin{bmatrix}\psi_1^2 & 0 & \psi_2^2 & \cdots & \psi_n^2 & 0 \\ 0 & \psi_1^2 & 0 & \psi_2^2 & \cdots & \psi_n^2\end{bmatrix}$$

$$\mathbf{S}^{\mathrm{T}} = \{S_1^x\ S_1^y\ S_2^x\ S_2^y \cdots S_n^x\ S_n^y\},\qquad \mathbf{w}^{\mathrm{T}} = \{w_1\ w_2\ w_3 \cdots w_n\}$$

(12.3.15)

and ψ_j^1 and ψ_j^2 are interpolation functions used for w and $(\phi_x,\ \phi_y)$, respectively. In general, ψ_j^1 and ψ_j^2 are polynomials of different degree. However, we take $\psi_j^1 = \psi_j^2 \equiv \psi_j$. This choice, as discussed for the Timoshenko beam element, requires the use of reduced integration for the evaluation of stiffness coefficients associated with the transverse shear strains.

Substituting Eq.(5.13) into Eqs.(5.7a) – (5.7c), we obtain the finite element model in expanded form

$$\begin{bmatrix} [M^{11}] & [0] & [0] \\ & [M^{22}] & [0] \\ \text{symmetric} & & [M^{33}] \end{bmatrix} \begin{Bmatrix} \ddot{w} \\ \ddot{S}^x \\ \ddot{S}^y \end{Bmatrix} + \begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ & [K^{22}] & [K^{23}] \\ \text{symmetric} & & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{S^x\} \\ \{S^y\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} \tag{5.16}$$

where

$$M_j^{11} = I_0 M_{ij}, \quad M_j^{22} = M_j^{33} = I_2 M_{ij}, \; M_{ij} = \int_{\Omega_e} \psi_i \psi_j \, dx\, dy$$

$$K_{ij}^{11} = \int_{\Omega_e} \left(A_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx\, dy$$

$$K_{ij}^{12} = \int_{\Omega_e} A_{55} \frac{\partial \psi_i}{\partial x} \psi_j \, dx\, dy$$

$$K_{ij}^{13} = \int_{\Omega_e} A_{44} \frac{\partial \psi_i}{\partial y} \psi_j \, dx\, dy$$

$$K_{ij}^{22} = \int_{\Omega_e} \left(D_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{55} \psi_i \psi_j \right) dx\, dy$$

$$K_{ij}^{23} = \int_{\Omega_e} \left(D_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dx\, dy$$

$$K_{ij}^{33} = \int_{\Omega_e} \left(D_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{44} \psi_i \psi_j \right) dx\, dy$$

$$F_i^1 = \int_{\Omega_e} q \psi_i \, dx\, dy + \oint_{\Gamma_e} Q_n \psi_i \, ds$$

$$F_i^2 = \oint_{\Gamma_e} \hat{M}_{nn} \psi_i \, ds, \quad F_i^3 = \oint_{\Gamma_e} \hat{M}_{ns} \psi_i \, ds \tag{5.17}$$

The vector of the finite element model is obtain by substituting Eq.(5.14) into Eq.(5.11):

$$\begin{bmatrix} \mathsf{M}^{11} & 0 \\ 0 & \mathsf{M}^{11} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{W}} \\ \ddot{\mathbf{S}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{W} \\ \mathbf{S} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{1} \\ \mathbf{F}^{2} \end{Bmatrix} \tag{5.18}$$

where [see Eqs.(5.12a) – (5.12b) for the definitions of **D, D**$_1$, etc.]

$$\mathsf{M}^{11} = \int_{\Omega_e} I_0 \Psi^1 (\Psi^1)^{\mathrm{T}} d\mathbf{x}, \quad \mathsf{M}^{22} = \int_{\Omega_e} I_2 (\Psi^2)^{\mathrm{T}} (\Psi^2) d\mathbf{x}$$

$$\mathbf{K}^{11} = \int_{\Omega_e} \mathbf{B}_1^{\mathrm{T}} \mathbf{A} \mathbf{B}_1 \, d\mathbf{x}, \quad \mathbf{K}^{12} = \int_{\Omega_e} \mathbf{B}_1^{\mathrm{T}} \mathbf{A} \Psi^2 d\mathbf{x} = \left(\mathbf{K}^{12}\right)^{\mathrm{T}}$$

$$\mathbf{K}^{22} = \int_{\Omega_e} \left[(\Psi^2)^{\mathrm{T}} \mathbf{A} \Psi^2 + \mathbf{B}^{\mathrm{T}} \mathbf{C} \mathbf{B} \right] d\mathbf{x}$$

$$\mathbf{F}^1 = \int_{\Omega_e} \Psi^1 q \, d\mathbf{x} + \oint_{\Gamma_e} \Psi^1 Q_n \, ds, \quad \mathbf{F}^2 = \oint_{\Gamma_e} (\Psi^2)^{\mathrm{T}} \mathbf{M}_{nn} \, ds$$

$$\mathbf{B}_1 = \mathbf{D}_1 (\Psi^1)^{\mathrm{T}} = \begin{bmatrix} \psi^1_{1,x} & \psi^1_{2,x} & \cdots & \psi^1_{n,x} \\ \psi^1_{1,y} & \psi^1_{2,y} & \cdots & \psi^1_{n,x} \end{bmatrix}_{(2\times n)}$$

$$\mathbf{B} = \mathbf{D}\Psi^2 = \begin{bmatrix} \psi^2_{1,x} & 0 & \psi^2_{2,x} & 0 & \cdots & \psi^2_{m,x} & 0 \\ 0 & \psi^2_{1,y} & 0 & \psi^2_{2,y} & \cdots & 0 & \psi^2_{m,y} \\ \psi^2_{1,y} & \psi^2_{1,x} & \psi^2_{2,y} & \psi^2_{2,x} & \cdots & \psi^2_{m,y} & \psi^2_{m,x} \end{bmatrix}_{(3\times 2m)} \tag{5.19}$$

The element Eq.(5.16) and Eq.(5.18) both can be written in compact form as

$$\mathsf{M}\ddot{\Delta} + \mathbf{K}\Delta = \mathbf{F} \tag{5.20}$$

where $\Delta = \{\mathbf{W}\mathbf{S}\}^{\mathrm{T}}$. The element stiffness matrix **K** and mass matrix **M** in Eq.(5.20) are of order $(n+2m)\times(n+2m)$, where *n* is the number of nodes per the Lagrange element use for *w* and *m* is the number of nodes per the Lagrange element used for ϕ_x and ϕ_y.

5.2 Magneto-hydrodynamics Double-diffusive Mixed Convection for Unsteady Flow

The governing equations for two-dimensional unsteady flow after invoking the Boussinesq approximation can be expressed as

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5.21}$$

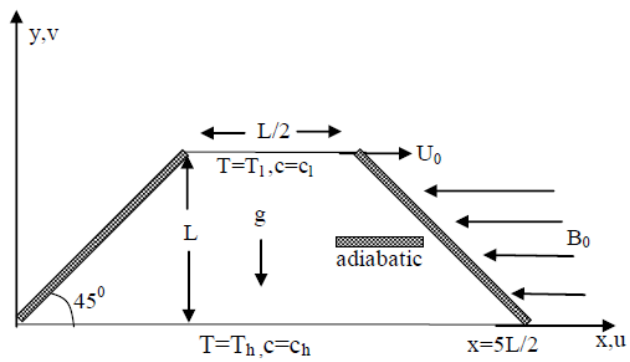


Fig.5.2: Schematic diagram for the problem with boundary conditions.

Momentum equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{5.22}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-g\beta_T(T-T_l)+g\beta_c(c-c_l)-\frac{\sigma B_0^2}{\rho}v \tag{5.23}$$

Energy equations

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{5.24}$$

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=D\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right) \tag{5.25}$$

where x and y are the distances measured along the horizontal and vertical directions respectively, u and v are the velocity components in the x and y are distances respectively, T denote the fluid temperature, T_i denotes the reference temperature for which buoyant force vanishes, p is the pressure and ρ is the fluid density, g is the gravitational constant, β_T and β_c are the volumetric coefficient of thermal and compositional expansion respectively, α and D are the thermal and mass diffusivity of fluid respectively.

The boundary and initial conditions of this physical problem are as follows

The initial conditions:

At $t = 0, u = v = 0, T = T_l, c = c_l$

The boundary conditions:

At the top wall: $u = U_0, v = 0, T = T_l, c = c_l$

At the inclined side walls: $u = v = 0, \frac{\partial T}{\partial \boldsymbol{n}} = 0, \frac{\partial c}{\partial \boldsymbol{n}} = 0$

At the bottom wall: $u = v = 0, T = T_h, c = c_h$

where $\boldsymbol{n}$ is the non-dimensional distance either along x or y direction acting normal to the surface.

Non-dimensional variables are used for making the governing Eqs.(5.21) – (5.25) into dimensionless form are as follows:

$$X = \frac{x}{L}, Y = \frac{y}{L}, U = \frac{u}{U_0}, V = \frac{v}{U_0}, P = \frac{(p+\rho g y)L^2}{\rho U_0^2},$$
$$\theta = \frac{T - T_l}{T_h - T_l}, C = \frac{c - c_l}{c_h - c_l}, \tau = \frac{U_0 t}{L}$$

where X and Y are the coordinates varying along horizontal and vertical directions, respectively, U and V are the velocity components in the X and Y directions, respectively, θ is the dimensionless temperature, C is the dimensionless concentration L is the height of the cavity and P is the dimensionless pressure. After substitution the dimensionless variables into the Eqs.(5.21) – (5.25) , we get the following dimensionless equations as

Continuity equation

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \tag{5.26}$$

Momentum equations

$$\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right) \tag{5.27}$$

$$\frac{\partial V}{\partial \tau}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-Ri(-\theta+BrC)-\frac{Ha^2}{\mathrm{Re}}V \tag{5.28}$$

Energy equations

$$\frac{\partial \theta}{\partial \tau}+U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}\right) \tag{5.29}$$

$$\frac{\partial C}{\partial \tau}+U\frac{\partial C}{\partial X}+V\frac{\partial C}{\partial Y}=\frac{1}{\mathrm{Re}\,\mathrm{Pr}\,Le}\left(\frac{\partial^2 C}{\partial X^2}+\frac{\partial^2 C}{\partial Y^2}\right) \tag{5.30}$$

The dimensionless parameters appearing in the Eqs.(5.26) – (5.30) are the Reynolds number (Re), Richardson number (*Ri*), Prandtl number (Pr), Buoyancy ratio (*Br*), Hartmann number (*Ha*) and Lewis number (*Le*). They are respectively defined as follows:

$$\mathrm{Re}=\frac{U_0 L}{v},\ Ri=\frac{g\beta_T(T_h-T_l)L}{U_0^2},\ Ha=\sqrt{\frac{\sigma}{\mu}}B_0 L$$

$$Br=\frac{\beta_c(c_h-c_l)}{\beta_h(T_h-T_l)},\ \mathrm{Pr}=\frac{v}{\alpha},\ Le=\frac{\alpha}{D}$$

The weighted residual process (Zienkiewicz and Taylor , 1991) is applied to drive the finite element equation to the Eqs.(5.21) – (5.25) as

$$\int N_\alpha\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dA=0$$

$$\int N_\alpha \left(\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} \right) dA = -\int H_\lambda \frac{\partial P}{\partial X} dA + \frac{1}{\text{Re}} \int N_\alpha \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) dA$$

$$\int N_\alpha \left(\frac{\partial V}{\partial \tau} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} \right) dA = -\int H_\lambda \frac{\partial P}{\partial Y} dA + \frac{1}{\text{Re}} \int N_\alpha \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) dA$$

$$-Ri\int N_\alpha \theta dA + RiBr\int N_\alpha C dA - \frac{Ha^2}{\text{Re}} \int N_\alpha V dA$$

$$\int N_\alpha \left(\frac{\partial \theta}{\partial \tau} + U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right) dA = \frac{1}{\text{Re}\Pr} \int N_\alpha \left(\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} \right) dA$$

$$\int N_\alpha \left(\frac{\partial C}{\partial \tau} + U\frac{\partial C}{\partial X} + V\frac{\partial C}{\partial Y} \right) dA = \frac{1}{\text{Re}\Pr Le} \int N_\alpha \left(\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} \right) dA$$

where A is the element area, $N_\alpha, \alpha = 1,2,3,\cdots,NP$ are the element shape functions or interpolation functions for the velocity components, temperature and concentration, and $H_\lambda, \lambda = 1,2,3,4$ are the element shape functions for the pressure.

The basic unknowns for the above differential equations are the velocity components *U*, *V,* the temperature, θ, concentration, *C* and the pressure, *P* are define below

$$U(X,Y) = N_\beta U_\beta,\ V(X,Y) = N_\beta V_\beta,\ \theta(X,Y) = N_\beta \theta_\beta,$$

$$C(X,Y) = N_\beta C_\beta,\ P(X,Y) = H_\mu P_\mu$$

where $\beta = 1,2,3,\cdots NP$ and $\mu = 1,2,3,4$.

Substituting the element velocity component distributions, the temperature distribution, the concentration distribution, and the pressure distribution from above equations and the finite element equations can be written in the form,

$$K_{\alpha\beta}^{x}U_{\beta}+K_{\alpha\beta}^{y}U_{\beta}=0$$

$$K_{\alpha\beta}\dot{U}_{\beta}+K_{\alpha\beta\gamma}^{x}U_{\beta}U_{\gamma}+K_{\alpha\beta\gamma}^{y}V_{\beta}U_{\gamma}+R_{\lambda\mu}^{x}P_{\mu}+\frac{1}{\mathrm{Re}}\left(K_{\alpha\beta}^{xx}+K_{\alpha\beta}^{yy}\right)U_{\beta}=Q_{\alpha}^{u}$$

$$K_{\alpha\beta}\dot{V}_{\beta}+K_{\alpha\beta\gamma}^{x}U_{\beta}V_{\gamma}+K_{\alpha\beta\gamma}^{y}V_{\beta}V_{\gamma}+R_{\lambda\mu}^{x}P_{\mu}+\frac{1}{\mathrm{Re}}\left(K_{\alpha\beta}^{xx}+K_{\alpha\beta}^{yy}\right)V_{\beta}+RiK_{\alpha\beta}\theta_{\beta}$$
$$-RiBrK_{\alpha\beta}C_{\beta}+\frac{Ha^{2}}{\mathrm{Re}}K_{\alpha\beta}V_{\beta}=Q_{\alpha}^{v}$$

$$K_{\alpha\beta}\dot{\theta}_{\beta}+K_{\alpha\beta\gamma}^{x}U_{\beta}\theta_{\gamma}+K_{\alpha\beta\gamma}^{y}V_{\beta}\theta_{\gamma}+\frac{1}{\mathrm{Re}}\left(K_{\alpha\beta}^{xx}+K_{\alpha\beta}^{yy}\right)\theta_{\beta}=Q_{\alpha}^{\theta}$$

$$K_{\alpha\beta}\dot{C}_{\beta}+K_{\alpha\beta\gamma}^{x}U_{\beta}C_{\gamma}+K_{\alpha\beta\gamma}^{y}V_{\beta}C_{\gamma}+\frac{1}{\mathrm{Re}}\left(K_{\alpha\beta}^{xx}+K_{\alpha\beta}^{yy}\right)C_{\beta}=Q_{\alpha}^{C}$$

where superposed dot denotes partial differentiation with respect to *t* and the coefficients in element matrices are in the form of the integrals over the element area and along the element edges S_0 (specifying surface tractions (Sx, Sy) along the inflow or outflow boundary) and S_w (concentration flux that flows into or out from the domain along wall boundary) as

$$K_{\alpha\beta}=\int N_{\alpha}N_{\beta}dA$$

$$K_{\alpha\beta}^{x}=\int N_{\alpha}\frac{\partial N_{\beta}}{\partial X}dA$$

$$K_{\alpha\beta}^{y}=\int N_{\alpha}\frac{\partial N_{\beta}}{\partial Y}dA$$

$$K_{\alpha\beta}^{xx}=\int\frac{\partial N_{\alpha}}{\partial X}\frac{\partial N_{\beta}}{\partial X}dA$$

$$K_{\alpha\beta}^{yy}=\int\frac{\partial N_{\alpha}}{\partial Y}\frac{\partial N_{\beta}}{\partial Y}dA$$

$$K_{\alpha\beta\gamma}^{x}=\int N_{\alpha}N_{\beta}\frac{\partial N_{\gamma}}{\partial X}dA$$

$$K^{y}_{\alpha\beta\gamma} = \int N_{\alpha} N_{\beta} \frac{\partial N_{\gamma}}{\partial Y} dA$$

$$R^{x}_{\lambda\mu} = \int N_{\lambda} \frac{\partial N_{\mu}}{\partial X} dA$$

$$R^{y}_{\lambda\mu} = \int N_{\lambda} \frac{\partial N_{\mu}}{\partial Y} dA$$

$$Q^{u}_{\alpha} = \frac{1}{\mathrm{Re}} \int N_{\alpha} S_{x} dS_{0}$$

$$Q^{v}_{\alpha} = \frac{1}{\mathrm{Re}} \int N_{\alpha} S_{y} dS_{0}$$

$$Q^{\theta}_{\alpha} = \frac{1}{\mathrm{Re}\,\mathrm{Pr}} \int N_{\alpha} q_{\theta} dS_{w}$$

$$Q^{c}_{\alpha} = \frac{1}{\mathrm{Re}\,\mathrm{Pr}\,Le} \int N_{\alpha} q_{c} dS_{w} \tag{5.31}$$

where $\alpha, \beta, \gamma = 1, 2, 3, \cdots NP$ and $\mu = 1, 2, 3, 4$

It is obvious to note that for the lower to higher order elements employing in the descretization a large number of integrals have to be evaluated. From Eq.(5.17) and Eq.(5.31) it is clear that there are thirteen types of integrals and among them five types of integrals are common for all the physical problems and the rest are problem dependent. Present literature indicates that those (thirteen types) integrals cannot be evaluated exactly by the most popular numerical integration technique such as Gauss quadrature rule (details can be seen in Karim, 2000-2003). Hence, immediate demand for the explicit schemes to evaluate such integrals are needed in the FEM solution procedure.

5.3 CONCLUSIONS

This chapter presented clearly all the integrals needed to form the element matrices for solving engineering problems employing the finite element method.

There are two classes of integrals of which five integrals are common for all the engineering problems and the rest of the integrals are problem dependent as it is found in the finite element formulations of the problems reviewed in this chapter. It is well known that the most important and the time consuming step in FEM solution procedure is to form all the element matrices exactly. In order to obtain the best accuracy of the evaluations of these integrals generally higher order Gaussian quadrature schemes are employed. That is the Gaussian quadrature with more Gaussian points and weights are needed to employ and hence the scheme increases the computing time. Therefore, it is an important task to make a proper balance between the accuracy and efficiency of calculations. It is astounding to note here that the explicit schemes for evaluating such integrals for the straight sided quadrilateral and curve (two straight and curved sides) triangles are available (Rathod and Karim, 2001, 2002). But, the implementation of such schemes are restricted only for the sub-parametric case only. Hence, it is a time demand to develop a technique for exact evaluation of such integrals to form all the element matrices in an efficient way. The next chapter will be fully concerned to present formulae for exact evaluation of such integrals.

Chapter 6

Derivation of Shape Functions in Global Coordinates and Exact Computation of Element Matrices for Quadrilateral Finite Elements

Chapter 6

Derivation of Shape Functions in Global Coordinates and Exact Computation of Element Matrices for Quadrilateral Finite Elements

This chapter includes a technique to derive shape functions in global co-ordinates for general quadrilateral finite elements. It identifies clearly the element geometry for which the shape functions in global co-ordinates cannot be derived and explains the way to overcome such situation. Finally, it presents formulae based on array multiplication for exact computation of different types of element matrices needed for the employment of the said element in finite element solution procedure. The computation process of element matrices require only: (1) the nodal co-ordinates of the element geometry to form a matrix G (say) and then it's inverse matrix H (say) and (2) the values of the integral of monomials over the element. All the components of element matrices are then computed by the product of components of H with the values of the integrals. Thus, the process reduces many time consuming steps of FEM solution procedure and that substantially reduces computational effort. The accuracy and efficiency of the formulae so presented are then demonstrated through application examples.

6.1 INTRODUCTION

Generally, for the convenience shape functions are commonly derived in local co-ordinates in local spaces. Transformation equations are written in terms of shape functions and the original element (in global space) is transformed into its contiguous element in local space. Consequently, all the calculations needed to form the element matrices that is the evaluation of numerous integrals are carried out in local co-ordinate systems (Zienkiewicz and Chaungy, 1965; Okabe, 1981a; Okabe, 1981b; Zienkiewicz and Morgan, 1983; Babu and Pinder, 1984; Reddy, 1984; Rathod, 1988; Hacker and Schreyer, 1989; Zienkiewicz and Taylor, 1989; Yagawa and Yashimara, 1990). It is well known that for such transformations (isoparametric, sub-parametric and super parametric) the integrals so encountered to form the stiffness matrix for the general quadrilateral (convex, concave) finite elements are rational integral of bivariate polynomial numerators with bilinear or higher order bivariate expressions denominators (Griffths, 1994; Griffths and Mustoe, 1995; Videla and Carsolaza, 1996; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001). Evaluation of such integrals defies our analytical skills and we are resort to numerical integration schemes (Zienkiewicz and Chaung, 1965; Zienkiewicz and Morgan, 1983; Barrett, 1999; Rathod and Karim, 2001). It is astounding to note here that the pleasing advancements in regard to analytical evaluation of such rational integrals for straight sided quadrilateral elements are made by many researchers (Okabe, 1981; Babu and Pinder, 1984; Rathod, 1988; Yagawa and Yashimara, 1990; Griffths, 1994; Griffths and Mustoe, 1995; Videla and Carsolaza, 1996; Rathod and Islam, 1998; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002). They have identified the drawbacks of numerical integration techniques especially for the Gaussian quadrature schemes. It is also evident from numerous research articles that such analytical integration formulae are applicable for sub-parametric case only and not applicable for higher order isoparametric elements. Besides that these analytical formulae require lot of computational effort and inconvenience for computer coding. Hence, for such difficulties and short comings the numerical integration schemes are still the only instrument for its simplicity and easy incorporation.

Among all the numerical integration schemes, Gaussian quadrature scheme occupies a central role for such evaluations. Complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric/ subparametric/ superparametric elements, the presence of the determinant of the Jacobean matrix in the denominator of the stiffness matrix for which the integrands are rational functions. Many authors (Yagawa and Yashimara, 1990; Lague and Baldur, 1997; Rathod and Islam, 1998; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001) outlined clearly that the usual Gaussian quadrature cannot evaluate exactly such integrals of rational functions as it can evaluate exactly a polynomial of degree $2n$-1 by employing n Gaussian points and weights. Obviously, for the desired accuracy of evaluations the number of Gaussian points and weights are needed to be increased and that increases substantially the computing time. Hence, a proper balance between the accuracy and efficiency is an important task (Yagawa and Yashimara, 1990; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002). Further, an attention is always required to select the order of the integrating rule as it is not yet totally worked out.

A suitable alternative, the use of polynomial shape functions in global coordinates other than of Dasgupta (2008) in the formulation give rise integrals of polynomials which can be exactly evaluated either by the selected order of the integrating rule or by analytical schemes. In this case the main barrier is the derivation of shape functions in global coordinates for the element under considerations. Especially it is very much difficult and so cumbersome in case of the quadrilateral elements. Considering all the facts and the popularity of the quadrilateral elements, we have concentrated to derive shape functions in global coordinates and to present all the components of element matrices in bivariate polynomial form in a systematic way. So that one can use the gaussian quadrature schemes or other numerical schemes easily for obtaining element matrices. The technique so implemented as: (1) formation of a matrix G (say) by the nodal co-ordinates of the element geometry and then its inverse matrix *H* (say), and (2) the values of the integral of monomials over the element. Finally, we have employed analytical schemes and presented all the components of element matrices as the expressions of products of components of matrix *H*. Thus, once the matrices *G*, *H* are formed and values of the integrals of monomials over the element are calculated the then computation of element matrices are computed by multiplication of components of matrix *H* only. It is clearly shown that the conventional derivation of polynomial shape functions in global co-ordinates and the other computations depends on non-singularity of matrix *G*. That is, if *G* is either singular or bad scaled then *H* cannot be computed and that leads impossibility of other computations. We have studied thoroughly and identified for the first time, two types of element geometry for which *G* is singular and shown geometrical reasons for which *G* is bad scaled. It has been found that slight change in the mesh may overcome such difficulties. One can easily apply the technique to derive polynomial shape functions in local co-ordinates by forming *G* matrix by local nodal co-ordinates.

The present technique reduces many time consuming steps of FEM solution procedure and that substantially reduces computational effort. For such substantiation, the Saint-Venant torsion problem studied by many researchers (Nguyen, 1992; Rathod and Islam, 1998; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002) is considered. The accuracy and efficiency of the presented formulae are then demonstrated through the calculation of Prandtl stress function values and torsional constant of different type of cross sections. Through comparison of computed results with the results of other researchers the importance of exact computation of element matrices are established. We believe that this study includes all the explicit expressions of element matrices in terms of components of matrix *H* which will be useful for solving other engineering problems. Therefore, we firmly believe that the technique of this basic study will be more contributing and attractive in the realm of application of the finite element method.

6.2 STRAIGHT SIDED QUADRILATERAL ELEMENTS

We consider here the straight sided 4-noned (Linear), 8-noded (quadratic) and 12-noded (cubic) serendipity quadrilateral elements as shown in Figs.6.1 – 6.3.

Usually, the field variable *u* (say) governing the physical problem is expressed as

$$u = \sum_{i=1}^{NP} N_i(x, y) u_i$$

where N_i is the shape function refer to the node *i* and *NP* is the number of points in the element and u_i is the functional values at node *i*.

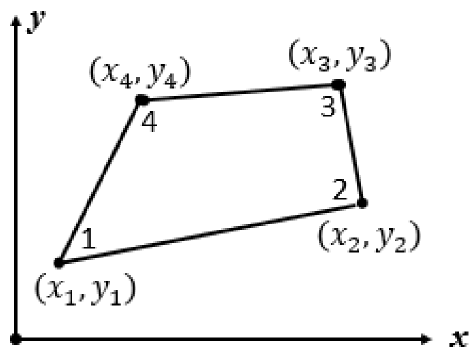


Fig.6.1: A 4-noded quadrilateral elements.

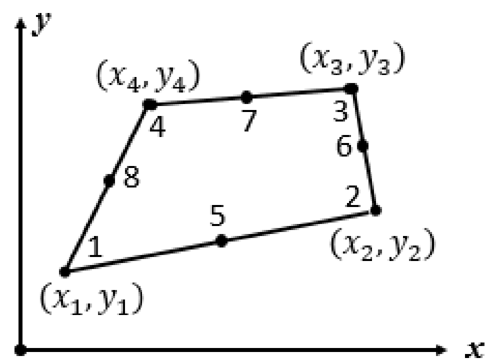


Fig.6.2: A 8-noded quadrilateral elements.

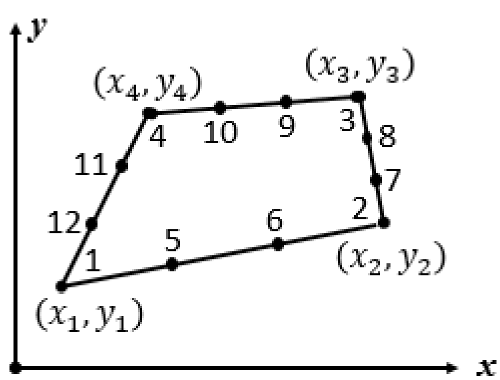


Fig.6.3: A 12-noded quadrilateral elements.

6.2.1 General form of shape functions

For convenience, we write shape functions for 4-noded, 8-noded and 12-noded quadrilateral elements respectively as

$$N_i(x,y)=a_i+b_ix+c_iy+d_ixy\,,\quad i=1,2,3,4 \tag{6.1}$$

$$N_i(x,y)=a_i+b_ix+c_iy+d_ixy+e_ix^2+f_iy^2+g_ix^2y+h_ixy^2,\, i=1,2,\cdots,8 \tag{6.2}$$

and

$$\begin{aligned}N_i(x,y)=a_i+b_ix+c_iy+d_ixy+e_ix^2+f_iy^2+g_ix^2y+h_ixy^2+l_ix^3\\+m_iy^3+n_ix^3y+p_ixy^3,\qquad i=1,2,\cdots,12\end{aligned} \tag{6.3}$$

where $a_i,b_i,c_i,\cdots,p_i$ are needed to determine by satisfying the properties of shape functions.

6.2.2 Derivation of shape functions

First, we consider the 4-noded quadrilateral element for deriving shape functions. By use of the properties

$$N_i(x_j,y_j)=\begin{cases}1 & \text{for } i=j\\ 0 & \text{for } i\neq j\end{cases}\quad ,i,j=1,2,3,4.$$

in Eq.(6.1), we have the following system of equations:

$$\begin{pmatrix}\delta_{i1}\\ \delta_{i2}\\ \delta_{i3}\\ \delta_{i4}\end{pmatrix}=\begin{pmatrix}1 & x_1 & y_1 & x_1y_1\\ 1 & x_2 & y_2 & x_2y_2\\ 1 & x_3 & y_3 & x_3y_3\\ 1 & x_4 & y_4 & x_4y_4\end{pmatrix}\begin{pmatrix}a_i\\ b_i\\ c_i\\ d_i\end{pmatrix}\text{ or, }\begin{pmatrix}\delta_{i1}\\ \delta_{i2}\\ \delta_{i3}\\ \delta_{i4}\end{pmatrix}=G\begin{pmatrix}a_i\\ b_i\\ c_i\\ d_i\end{pmatrix}$$

$$\text{or, }\begin{pmatrix}a_i\\ b_i\\ c_i\\ d_i\end{pmatrix}=G^{-1}\begin{pmatrix}\delta_{i1}\\ \delta_{i2}\\ \delta_{i3}\\ \delta_{i4}\end{pmatrix} \tag{6.1.1}$$

where $G = \begin{pmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{pmatrix}$ and croneker delta, $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

If we assume $H = G^{-1}$, then Eq.(6.1.1) becomes

$$\begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \mathrm{H} \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{pmatrix}$$

Then we obtain $a_i = H_{1i}, b_i = H_{2i}, c_i = H_{3i}, d_i = H_{4i}\ , \quad i = 1,2,3,4$

Finally, Eq.(6.1) is expressed as

$$N_i(x,y) = \mathrm{H}_{1i} + \mathrm{H}_{2i}x + \mathrm{H}_{3i}y + \mathrm{H}_{4i}xy, \quad i = 1,2,3,4 \tag{6.4}$$

Proceeding in the similar way, we obtain shape functions for 8-noded and 12-noded quadrilateral elements respectively as:

$$\begin{aligned} N_i(x,y) = {} & \mathrm{H}_{1i} + \mathrm{H}_{2i}x + \mathrm{H}_{3i}y + \mathrm{H}_{4i}xy \\ & + \mathrm{H}_{5i}x^2 + \mathrm{H}_{6i}y^2 + \mathrm{H}_{7i}x^2y + \mathrm{H}_{8i}xy^2\ , \quad i = 1,2,\cdots,8 \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} N_i(\mathrm{x,y}) = {} & \mathrm{H}_{1i} + \mathrm{H}_{2i}x + \mathrm{H}_{3i}y + \mathrm{H}_{4i}xy + \mathrm{H}_{5i}x^2 + \mathrm{H}_{6i}y^2 + \mathrm{H}_{7i}x^2y + \mathrm{H}_{8i}xy^2 \\ & + \mathrm{H}_{9i}x^3 + \mathrm{H}_{10i}y^3 + \mathrm{H}_{11i}x^3y + \mathrm{H}_{12i}xy^3, \ i = 1,2,\cdots,12 \end{aligned} \tag{6.6}$$

Note here that all the shape functions are now explicitly written by the components of matrix $H = G^{-1}$ and it is verified that $\sum_{i=1}^{NP} N_i(x,y) = 1$ i.e. the completeness property is satisfied.

6.2.3 Algorithm to form *G* matrix

By the known nodal co-ordinates $(x_i, y_i), i = 1, 2, 3, \cdots, NP$ the matrix *G* easily can be formed by the algorithm:

for i = 1,2,3, … , NP

G_{i1} = 1; $G_{i2} = x_i$; $G_{i2} = y_i$; $G_{i3} = x_i\, y_i$;

If ((NP == 8) or (NP == 12)) then

$G_{i5} = x_i^2;\ G_{i6} = y_i^2;\ G_{i7} = x_i^3 y_i; G_{i8} = x_i y_i^3;$

If (NP ==12)) then

$G_{i9} = x_i^3;\ G_{i10} = y_i^3;\ G_{i7} = x_i^2 y_i; G_{i8} = x_i y_i^2;$

end if

end if

6.3 SINGULARITY AND NON-SINGULARITY CONDITIONS FOR *G* MATRIX

Through Eqs.(6.4) – (6.6), it is clear that all the shape functions $N_i(x, y)$ may be derived if G is invertible i.e. $H = G^{-1}$ is computed. That is if *G* is non-singular then its inverse matrix *H* can be computed and shape functions can be derived. Otherwise the shape functions cannot be derived.

Therefore, it is an important task to investigate throughly the singularity of *G* formed by the nodal co-ordinates of the element geometry.

For detailed study, we consider first the 4-noded quadrilateral element for which

$$G = \begin{pmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{pmatrix}$$

Case-1: If $x_1 = x_3$ and $y_2 = y_4$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.4 then $|G| = 0$ that is G is singular.

Case-2: If $x_2 = x_4$ and $y_1 = y_3$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.5 then $|G| = 0$ that is G is singular.

Case-3: If $y_1 = y_3$ and $x_2 \neq x_4$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.6 then $|G| \neq 0$ that is G is non-singular.

Case-4: If $x_1 = x_3$ and $y_2 \neq y_4$ for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.7 then $|G| \neq 0$ that is G is non-singular.

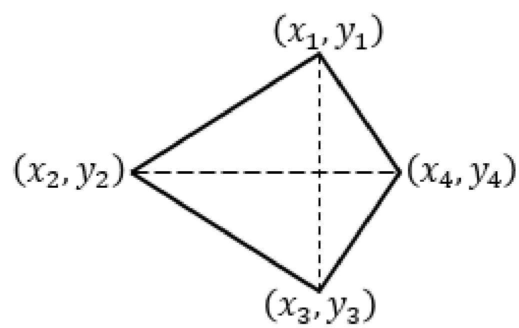


Fig.6.4: $|G| = 0$ for the quadrilateral when $x_1 = x_3$ and $y_2 = y_4$.

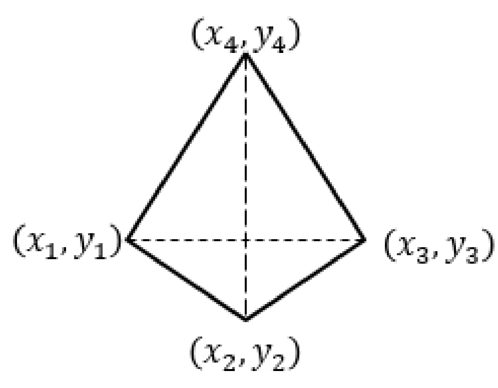


Fig. 6.5: $|G| = 0$ for the quadrilateral when $x_2 = x_4$ and $y_1 = y_3$.

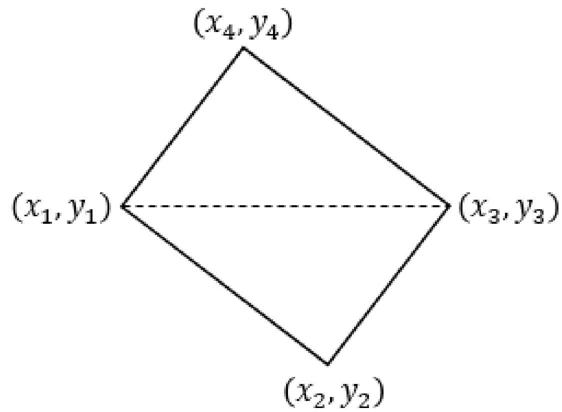


Fig. 6.6: $|G| \neq 0$ for the quadrilateral when $y_1 = y_3$ and $x_2 \neq x_4$.

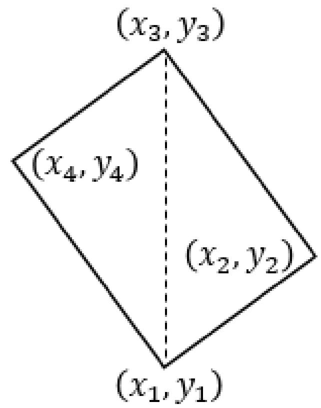


Fig.6.7: $|G| \neq 0$ for the quadrilateral when $x_1 = x_3$ and $y_2 \neq y_4$.

Case-5: If $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 = y_4$ (for rectangular shape) for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.8 then $|G| \neq 0$ that is G is non-singular.

Case-6: If $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 \neq y_4$ (for trapezoidal shape) for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.9 then $|G| \neq 0$ that is G is non-singular.

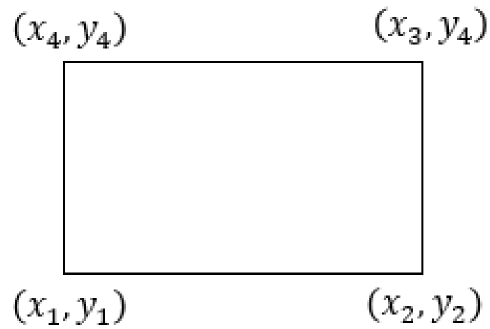


Fig.6.8: $|G| \neq 0$ for the quadrilateral when $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 = y_4$.

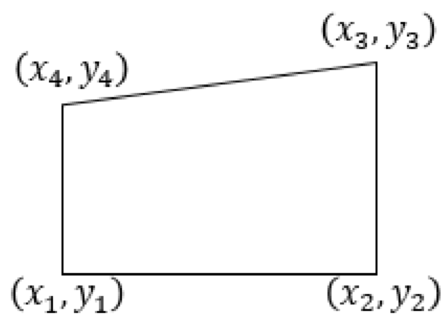


Fig.6.9: $|G| \neq 0$ for the quadrilateral when $x_1 = x_4, x_2 = x_3, y_1 = y_2, y_3 \neq y_4$.

Case-7: If $x_1 = x_4, x_2 \neq x_3, y_1 = y_2, y_3 = y_4$ (for other trapezoidal shape) for coordinates $(x_i, y_i), i = 1,2,3,4$ of a quadrilateral as shown in Fig.6.10 then $|G| \neq 0$ that is G is non-singular.

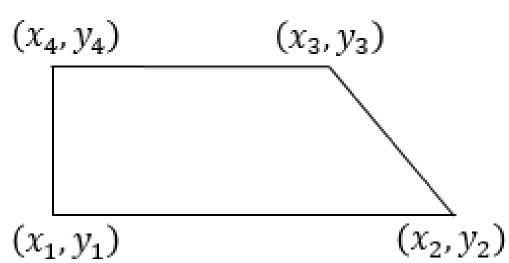


Fig.6.10: $|G| \neq 0$ for the quadrilateral when $x_1 = x_4, x_2 \neq x_3, y_1 = y_2, y_3 = y_4$.

Therefore, it can be concluded that only in two cases (case 1 and case 2) G is singular for which shape functions cannot be derived. But, in other cases G is non-singular and hence shape functions can be derived. For the case $|G| = 0$ (case-1, case-2) changes in mesh is necessary for which all the quadrilaterals are different from the quadrilaterals shown in Figs.6.4 – 6.5. More specifically, we can overcome the situation of singularity of G by changing co-ordinates of the quadrilaterals by remashing the domain of the problem.

Similarly, G matrix for 8-noded and 12-noded quadrilaterals are analyzed and it is found that for all straight sided element geometry G is always non-singular and hence shape functions can be derived. But if element contains curved side, then (*i*) $|G| \neq 0$ for 8-noded quadrilaterals, and (*ii*) $|G| = 0$ or G may be bad scaled for 12-noded quadrilaterals. For example, if we consider OAB of Fig.6.11 as one 12-noded (single) element then G is singular. Whereas for the models as shown in Fig.6.11.2 (a)-(b), G is non-singular for all 12-noded quadrilaterals.

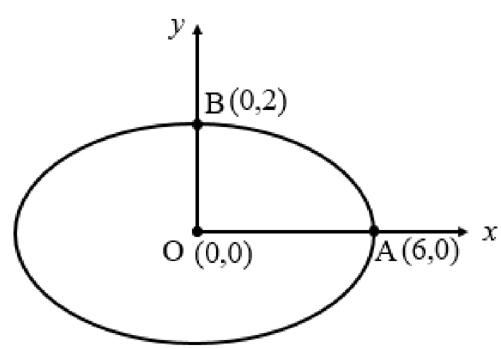


Fig.6.11: An elliptic cross section.

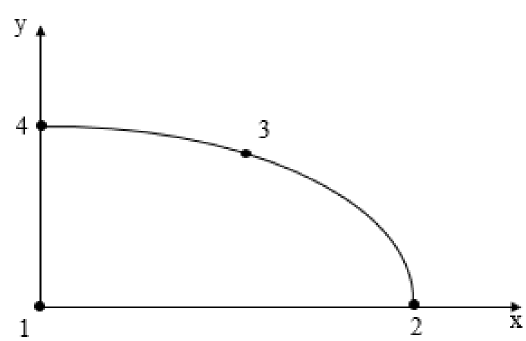


Fig.6.11.1(a): Finite Element Model-1 with one 4 node quadrilateral elements.

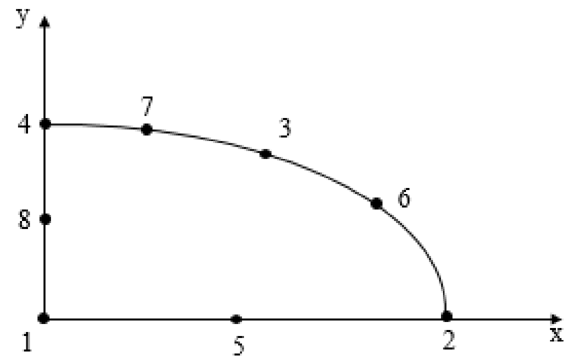


Fig.6.11.1(b): Finite Element Model-2 with one 8 node quadrilateral elements.

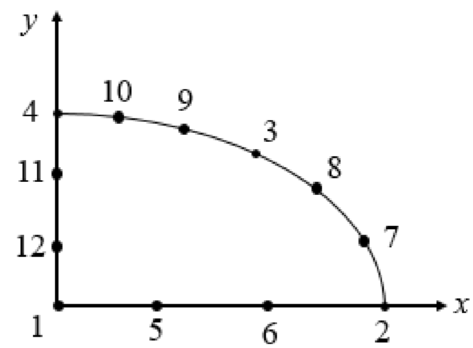


Fig.6.11.1(c): Finite Element Model-3 with one 12 node quadrilateral elements.

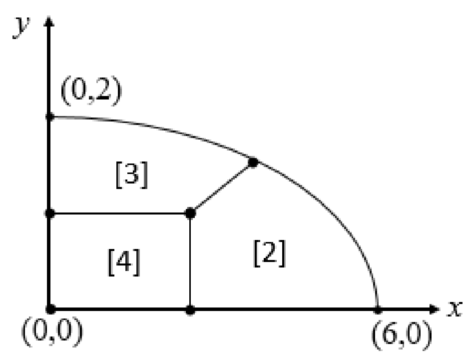


Fig.6.11.2(a): Finite Element Model-4 with three 4 node quadrilateral elements.

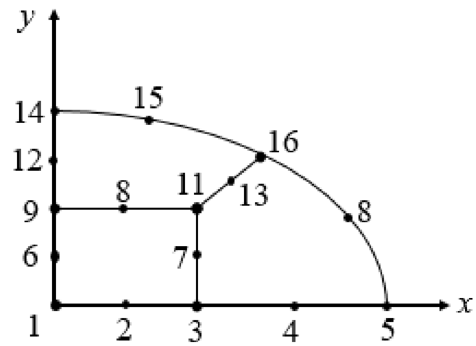


Fig.6.11.2(b): Finite Element Model-5with three 8 node quadrilateral elements.

We have investigated thoroughly the cases for which *G* is either singular or non-singular and it is found that *G* is always non-singular for 8-noded quadrilaterals.

6.4 GLOBAL DERIVATIVES, PRODUCT OF GLOBAL DERIVATIVES AND COMPONENTS OF ELEMENT MATRICES

This section deals with global derivatives, product of global derivatives and computation of various components of element matrices for quadrilateral finite elements. Step by step calculation process is shown for 4-noded quadrilateral element in details. Similar process is carried out for 8-noded and 12-noded (serendipity) quadrilateral elements. One can follow the process also for Lagrange type higher order quadrilateral elements.

For 4-noded quadrilateral element, using Eq.(6.4), we obtain

$$\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} = \mathrm{H}_{2\mathrm{i}}\mathrm{H}_{2\mathrm{j}} + (\mathrm{H}_{2\mathrm{j}}\mathrm{H}_{4\mathrm{i}} + \mathrm{H}_{2\mathrm{i}}\mathrm{H}_{4\mathrm{j}})y + \mathrm{H}_{4\mathrm{i}}\mathrm{H}_{4\mathrm{j}}y^2 \qquad (6.6)$$

6.4.1 Integration formula for $I^{\alpha,\beta}$

It is necessary in FEM to integrate the product of shape functions, product of derivatives and other type of products e.g. product of shape functions and their derivatives over the element geometry, (Reddy, 1984). Since, such products are now in polynomial form one can exactly evaluate all these integrals by use of Gaussian quadrature schemes. Here, we intended to employ the following analytical integration formula.

Integral formulae: If Γ is a polygonal boundary of a quadrilateral enclosed by the vertices $(x_i, y_i), i = 1,2,3,\cdots,NP$ with $x_{NP+1} = x_{NP}$, $y_{NP+1} = y_{NP}$. Then the integral of monomials over the quadrilateral i.e. $I^{\alpha,\beta} = \iint_{\partial} x^{\alpha} y^{\beta} dxdy$, where ∂ is the domain enclosed by Γ can be expressed as:

$$I^{\alpha,\beta} = \frac{1}{\alpha+1}\sum_{i=1}^{NP}\left[\sum_{p=0}^{\alpha+1}\sum_{q=0}^{\beta}\frac{\binom{\alpha+1}{p}\binom{\beta}{q}}{(p+q+1)} x_i^{\alpha+1-p} X_i^{p} y_i^{\beta-p} Y_i^{q+1}\right] \tag{5.4.2}$$

where α, β are non-negative positive integers and $X_i = x_{i+1} - x_i$, $Y_i = y_{i+1} - y_i$.

6.4.2 Integration of bivariate polynomials to compute element matrices of 4-noded quadrilateral

Integrating Eq.(5.4.1) by use of Eq.(5.4.2), we have one type of components of element matrices

$$K_{ij}^{xx} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dxdy = \mathrm{H}_{2i}\mathrm{H}_{2j} I^{0,0} + (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j}) I^{0,1} + \mathrm{H}_{4i}\mathrm{H}_{4j} I^{0,2}$$

Similarly, we have the other components of element matrices as in the following:

$$K_{ij}^{yy} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} dxdy = \mathrm{H}_{3i}\mathrm{H}_{3j} I^{0,0} + (\mathrm{H}_{3j}\mathrm{H}_{4i} + \mathrm{H}_{3i}\mathrm{H}_{4j}) I^{1,0} + \mathrm{H}_{4i}\mathrm{H}_{4j} I^{2,0}$$

$$K_{ij}^{xy} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} dxdy = \mathrm{H}_{2i}\mathrm{H}_{3j} I^{0,0} + \mathrm{H}_{2i}\mathrm{H}_{4j} I^{1,0} + \mathrm{H}_{3j}\mathrm{H}_{4i} I^{0,1} + \mathrm{H}_{4i}\mathrm{H}_{4j} I^{1,1}$$

$$K_{ij}^{x} = \iint N_i(x,y)\frac{\partial}{\partial x}(N_j(x,y))dxdy$$
$$= H_{1i}H_{2j}I^{0,0} + H_{2i}H_{2j}I^{1,0} + (H_{2j}H_{3i} + H_{1i}H_{4j})I^{0,1}$$
$$+ (H_{2j}H_{4i} + H_{2i}H_{4j})I^{1,1} + H_{3i}H_{4j}I^{0,2} + H_{4i}H_{4j}I^{1,2}$$

$$K_{ij}^{y} = \iint N_i(x,y)\frac{\partial}{\partial y}(N_j(x,y))dxdy$$
$$= H_{1i}H_{3j}I^{0,0} + (H_{2i}H_{3j} + H_{1i}H_{4j})I^{1,0} + H_{2i}H_{4j}I^{2,0}$$
$$+ H_{3i}H_{3j}I^{0,1} + (H_{3j}H_{4i} + H_{3i}H_{4j})I^{1,1} + H_{4i}H_{4j}I^{2,1}$$

$$F_i = \int N_i(x,y)dxdy = H_{1i}I^{0,0} + H_{2i}I^{1,0} + H_{3i}I^{0,1} + H_{4i}I^{1,1}$$

$$B_{ij} = \int N_i(x,y)N_j(x,y)dxdy$$
$$= H_{1i}H_{1j}I^{0,0} + (H_{1j}H_{2i} + H_{1i}H_{2j})I^{1,0} + H_{2i}H_{2j}I^{2,0} + (H_{1j}H_{3i} + H_{1i}H_{3j})I^{0,1}$$
$$+ (H_{2j}H_{3i} + H_{2i}H_{3j} + H_{1j}H_{4i} + H_{1i}H_{4j})I^{1,1} + (H_{2j}H_{4i} + H_{2i}H_{4j})I^{2,1}$$
$$+ H_{3i}H_{3j}I^{0,2} + (H_{3j}H_{4i} + H_{3i}H_{4j})I^{1,2} + H_{4i}H_{4j}I^{2,2}$$

$$R_{ijk}^{x} = \iint N_i(\mathrm{x},\mathrm{y})N_j(x,y)\frac{\partial N_k}{\partial x}dxdy$$
$$= H_{1i}H_{1j}H_{2k}I^{0'0} + (H_{1j}H_{2i}H_{2k} + H_{1i}H_{2j}H_{2k})I^{1,0} + H_{2i}H_{2j}H_{2k}I^{2,0}$$
$$+ (H_{1j}H_{2k}H_{3i} + H_{1i}H_{2k}H_{3j} + H_{1i}H_{1j}H_{4k})I^{0,1} + (H_{2j}H_{2k}H_{3i} + H_{2i}H_{2k}H_{3j} + H_{1j}H_{2k}H_{4i}$$
$$+ H_{1i}H_{2k}H_{4j} + H_{1j}H_{2i}H_{4k} + H_{1i}H_{2j}H_{4k})I^{1,1} + (H_{2j}H_{2k}H_{4i} + H_{2i}H_{2k}H_{4j} + H_{2i}H_{2j}H_{4k})I^{2,1}$$
$$+ (H_{2k}H_{3i}H_{3j} + H_{1j}H_{3i}H_{4k} + H_{1i}H_{3j}H_{4k})I^{0,2} + (H_{2k}H_{3j}H_{4i} + H_{2k}H_{3i}H_{4j} + H_{2j}H_{3i}H_{4k}$$
$$+ H_{2i}H_{3j}H_{4k} + H_{1j}H_{4i}H_{4k} + H_{1i}H_{4j}H_{4k})I^{1,2} + (H_{2k}H_{4i}H_{4j} + H_{2j}H_{4i}H_{4k} + H_{2i}H_{4j}H_{4k})I^{2,2}$$
$$+ H_{3i}H_{3j}H_{4k}I^{0,3} + (H_{3j}H_{4i}H_{4k} + H_{3i}H_{4j}H_{4k})I^{1,3} + H_{4i}H_{4j}H_{4k}I^{2,3}$$

$$R_{ijk}^{y}=\iint N_i(\mathrm{x},\mathrm{y})N_j(x,y)\frac{\partial N_k}{\partial y}dxdy$$
$$=\mathrm{H}_{1i}\mathrm{H}_{1j}\mathrm{H}_{3k}I^{0,0}+(\mathrm{H}_{1j}\mathrm{H}_{2i}\mathrm{H}_{3k}+\mathrm{H}_{1i}\mathrm{H}_{2j}\mathrm{H}_{3k}+\mathrm{H}_{1i}\mathrm{H}_{1j}\mathrm{H}_{4k})I^{1,0}+(\mathrm{H}_{2i}\mathrm{H}_{2j}\mathrm{H}_{3k}$$
$$+\mathrm{H}_{1j}\mathrm{H}_{2i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{2j}\mathrm{H}_{4k})I^{2,0}+\mathrm{H}_{2i}\mathrm{H}_{2j}\mathrm{H}_{4k}I^{3,0}+(\mathrm{H}_{1j}\mathrm{H}_{3i}\mathrm{H}_{3k}+\mathrm{H}_{1i}\mathrm{H}_{3j}\mathrm{H}_{3k})I^{0,1}$$
$$+(\mathrm{H}_{2j}\mathrm{H}_{3i}\mathrm{H}_{3k}+\mathrm{H}_{2i}\mathrm{H}_{3j}\mathrm{H}_{3k}+\mathrm{H}_{1j}\mathrm{H}_{3k}\mathrm{H}_{4i}+\mathrm{H}_{1i}\mathrm{H}_{3k}\mathrm{H}_{4j}+\mathrm{H}_{1j}\mathrm{H}_{3i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{3j}\mathrm{H}_{4k})I^{1,1}$$
$$+(\mathrm{H}_{2j}\mathrm{H}_{3k}\mathrm{H}_{4i}+\mathrm{H}_{2i}\mathrm{H}_{3k}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{3i}\mathrm{H}_{4k}+\mathrm{H}_{2i}\mathrm{H}_{3j}\mathrm{H}_{4k}+\mathrm{H}_{1j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{1i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{2,1}$$
$$+(\mathrm{H}_{2j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{2i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{3,1}+\mathrm{H}_{3i}\mathrm{H}_{3j}\mathrm{H}_{3k}I^{0,2}+(\mathrm{H}_{3j}\mathrm{H}_{3k}\mathrm{H}_{4i}+\mathrm{H}_{3i}\mathrm{H}_{3k}\mathrm{H}_{4j}$$
$$+\mathrm{H}_{3i}\mathrm{H}_{3j}\mathrm{H}_{4k})I^{1,2}+(\mathrm{H}_{3k}\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{3j}\mathrm{H}_{4i}\mathrm{H}_{4k}+\mathrm{H}_{3i}\mathrm{H}_{4j}\mathrm{H}_{4k})I^{2,2}+\mathrm{H}_{4i}\mathrm{H}_{4j}\mathrm{H}_{4k}I^{3,2}$$

Formulae for other four integrals:

$$Q_\alpha^x=\int N_\alpha S_x\, dS_\theta,\quad Q_\alpha^y=\int N_\alpha S_y\, dS_\theta,\quad Q_\alpha^\theta=\int N_\alpha S_\theta\, dS_w,\quad Q_\alpha^c=\int N_\alpha S_c\, dS_w$$

can be derived as on the same line of the formula for $F_i=\int N_i\, dxdy$.

Proceeding on the similar way, we obtain component of element matrices for 8-noded and 12-noded quadrilateral elements as the following:

6.4.3 Element matrices for 8-noded quadrilateral elements

$$K_{ij}^{xx}=\mathrm{H}_{2i}\mathrm{H}_{2j}I^{0,0}+2(\mathrm{H}_{2j}\mathrm{H}_{5i}+\mathrm{H}_{2i}\mathrm{H}_{5j})I^{1,0}+4\mathrm{H}_{5i}\mathrm{H}_{5j}I^{2,0}+(\mathrm{H}_{2j}\mathrm{H}_{4i}+\mathrm{H}_{2i}\mathrm{H}_{4j})I^{0,1}$$
$$+2(\mathrm{H}_{4j}\mathrm{H}_{5i}+\mathrm{H}_{4i}\mathrm{H}_{5j}+\mathrm{H}_{2j}\mathrm{H}_{7i}+\mathrm{H}_{2i}\mathrm{H}_{7j})I^{1,1}+4(\mathrm{H}_{5j}\mathrm{H}_{7i}+\mathrm{H}_{5i}\mathrm{H}_{7j})I^{2,1}$$
$$+(\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{2j}\mathrm{H}_{8i}+\mathrm{H}_{2i}\mathrm{H}_{8j})I^{0,2}+2(\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j}+\mathrm{H}_{5j}\mathrm{H}_{8i}+\mathrm{H}_{5i}\mathrm{H}_{8j})I^{1,2}$$
$$+4\mathrm{H}_{7i}\mathrm{H}_{7j}I^{2,2}+(\mathrm{H}_{4j}\mathrm{H}_{8i}+\mathrm{H}_{4i}\mathrm{H}_{8j})I^{0,3}+2(\mathrm{H}_{7j}\mathrm{H}_{8i}+\mathrm{H}_{7i}\mathrm{H}_{8j})I^{1,3}+\mathrm{H}_{8i}\mathrm{H}_{8j}I^{0,4}$$

$$K_{ij}^{yy}=\mathrm{H}_{3i}\mathrm{H}_{3j}I^{0,0}+(\mathrm{H}_{3j}\mathrm{H}_{4i}+\mathrm{H}_{3i}\mathrm{H}_{4j})I^{1,0}+(\mathrm{H}_{4i}\mathrm{H}_{4j}+\mathrm{H}_{3j}\mathrm{H}_{7i}+\mathrm{H}_{3i}\mathrm{H}_{7j})I^{2,0}$$
$$+(\mathrm{H}_{4j}\mathrm{H}_{7i}+\mathrm{H}_{4i}\mathrm{H}_{7j})I^{3,0}+\mathrm{H}_{7i}\mathrm{H}_{7j}I^{4,0}+2(\mathrm{H}_{3j}\mathrm{H}_{6i}+\mathrm{H}_{3i}\mathrm{H}_{6j})I^{0,1}$$
$$+2(\mathrm{H}_{4j}\mathrm{H}_{6i}+\mathrm{H}_{4i}\mathrm{H}_{6j}+\mathrm{H}_{3j}\mathrm{H}_{8i}+\mathrm{H}_{3i}\mathrm{H}_{8j})I^{1,1}+2(\mathrm{H}_{6j}\mathrm{H}_{7i}+\mathrm{H}_{6i}\mathrm{H}_{7j}$$
$$+\mathrm{H}_{4j}\mathrm{H}_{8i}+\mathrm{H}_{4i}\mathrm{H}_{8j})I^{2,1}+2(\mathrm{H}_{7j}\mathrm{H}_{8i}+\mathrm{H}_{7i}\mathrm{H}_{8j})I^{3,1}+4\mathrm{H}_{6i}\mathrm{H}_{6j}I^{0,2}$$
$$+4(\mathrm{H}_{6j}\mathrm{H}_{8i}+\mathrm{H}_{6i}\mathrm{H}_{8j})I^{1,2}+4\mathrm{H}_{8i}\mathrm{H}_{8j}I^{2,2}$$

$$K_{ij}^{xy} = \mathrm{H}_{2i}\mathrm{H}_{3j}I^{0,0} + (\mathrm{H}_{2i}\mathrm{H}_{4j} + 2\mathrm{H}_{3j}\mathrm{H}_{5i})I^{1,0} + (2\mathrm{H}_{4j}\mathrm{H}_{5i} + \mathrm{H}_{2i}\mathrm{H}_{7j})I^{2,0}$$
$$+ (\mathrm{H}_{3j}\mathrm{H}_{4i} + 2\mathrm{H}_{2i}\mathrm{H}_{6j})I^{0,1} + (\mathrm{H}_{4i}\mathrm{H}_{4j} + 4\mathrm{H}_{5i}\mathrm{H}_{6j} + 2\mathrm{H}_{3j}\mathrm{H}_{7i} + 2\mathrm{H}_{2i}\mathrm{H}_{8j})I^{1,1}$$
$$+ (2\mathrm{H}_{4j}\mathrm{H}_{7i} + \mathrm{H}_{4i}\mathrm{H}_{7j} + 4\mathrm{H}_{5i}\mathrm{H}_{8j})I^{2,1} + 2\mathrm{H}_{7i}\mathrm{H}_{7j}I^{3,1} + (2\mathrm{H}_{4i}\mathrm{H}_{6j} + \mathrm{H}_{3j}\mathrm{H}_{8i})I^{0,2}$$
$$+ (4\mathrm{H}_{6j}\mathrm{H}_{7i} + \mathrm{H}_{4j}\mathrm{H}_{8i} + 2\mathrm{H}_{4i}\mathrm{H}_{8j})I^{1,2} + (\mathrm{H}_{7j}\mathrm{H}_{8i} + 4\mathrm{H}_{7i}\mathrm{H}_{8j})I^{2,2}$$
$$+ 2\mathrm{H}_{8i}\mathrm{H}_{8j}I^{1,3} + 2\mathrm{H}_{5i}\mathrm{H}_{7j}I^{,03} + 2\mathrm{H}_{6j}\mathrm{H}_{8i}I^{0,3}$$

$$K_{ij}^{x} = \mathrm{H}_{1i}\mathrm{H}_{2j}I^{0,0} + (\mathrm{H}_{2i}\mathrm{H}_{2j} + 2\mathrm{H}_{1i}\mathrm{H}_{5j})I^{1,0} + (\mathrm{H}_{2j}\mathrm{H}_{5i} + 2\mathrm{H}_{2i}\mathrm{H}_{5j})I^{2,0} + 2\mathrm{H}_{5i}\mathrm{H}_{5j}I^{3,0}$$
$$+ (\mathrm{H}_{2j}\mathrm{H}_{3i} + \mathrm{H}_{1i}\mathrm{H}_{4j})I^{0,1} + (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j} + 2\mathrm{H}_{3i}\mathrm{H}_{5j} + 2\mathrm{H}_{1i}\mathrm{H}_{7j})I^{1,1}$$
$$+ (\mathrm{H}_{4j}\mathrm{H}_{5i} + 2\mathrm{H}_{4i}\mathrm{H}_{5j} + \mathrm{H}_{2j}\mathrm{H}_{7i} + 2\mathrm{H}_{2i}\mathrm{H}_{7j})I^{2,1} + 2(\mathrm{H}_{5j}\mathrm{H}_{7i} + \mathrm{H}_{5i}\mathrm{H}_{7j})I^{3,1}$$
$$+ (\mathrm{H}_{3i}\mathrm{H}_{4j} + \mathrm{H}_{2j}\mathrm{H}_{6i} + \mathrm{H}_{1i}\mathrm{H}_{8j})I^{0,2} + (\mathrm{H}_{4i}\mathrm{H}_{4j} + 2\mathrm{H}_{5j}\mathrm{H}_{6i} + 2\mathrm{H}_{3i}\mathrm{H}_{7j} + \mathrm{H}_{2j}\mathrm{H}_{8i}$$
$$+ \mathrm{H}_{2i}\mathrm{H}_{8j})I^{1,2} + (\mathrm{H}_{4j}\mathrm{H}_{7i} + 2\mathrm{H}_{4i}\mathrm{H}_{7j} + 2\mathrm{H}_{5j}\mathrm{H}_{8i} + \mathrm{H}_{5i}\mathrm{H}_{8j})I^{2,2} + 2\mathrm{H}_{7i}\mathrm{H}_{7j}I^{3,2}$$
$$+ (\mathrm{H}_{4j}\mathrm{H}_{6i} + \mathrm{H}_{3i}\mathrm{H}_{8j})I^{0,3} + (2\mathrm{H}_{6i}\mathrm{H}_{7j} + \mathrm{H}_{4j}\mathrm{H}_{8i} + \mathrm{H}_{4i}\mathrm{H}_{8j})I^{1,3} + (2\mathrm{H}_{7j}\mathrm{H}_{8i}$$
$$+ \mathrm{H}_{7i}\mathrm{H}_{8j})I^{2,3} + \mathrm{H}_{6i}\mathrm{H}_{8j}I^{0,4} + \mathrm{H}_{8i}\mathrm{H}_{8j}I^{1,4}$$

$$K_{ij}^{y} = \mathrm{H}_{1i}\mathrm{H}_{3j}I^{0,0} + (\mathrm{H}_{2i}\mathrm{H}_{3j} + \mathrm{H}_{1i}\mathrm{H}_{4j})I^{1,0} + (\mathrm{H}_{2i}\mathrm{H}_{4j} + \mathrm{H}_{3j}\mathrm{H}_{5i} + \mathrm{H}_{1i}\mathrm{H}_{7j})I^{2,0} + (\mathrm{H}_{4j}\mathrm{H}_{5i}$$
$$+ \mathrm{H}_{2i}\mathrm{H}_{7j})I^{3,0} + \mathrm{H}_{5i}\mathrm{H}_{7j}I^{4,0} + (\mathrm{H}_{3i}\mathrm{H}_{3j} + 2\mathrm{H}_{1i}\mathrm{H}_{6j})I^{0,1} + (\mathrm{H}_{3j}\mathrm{H}_{4i} + \mathrm{H}_{3i}\mathrm{H}_{4j} + 2\mathrm{H}_{2i}\mathrm{H}_{6j}$$
$$+ 2\mathrm{H}_{1i}\mathrm{H}_{8j})I^{1,1} + (\mathrm{H}_{4i}\mathrm{H}_{4j} + 2\mathrm{H}_{5i}\mathrm{H}_{6j} + \mathrm{H}_{3j}\mathrm{H}_{7i} + \mathrm{H}_{3i}\mathrm{H}_{7j} + 2\mathrm{H}_{2i}\mathrm{H8j})I^{2,1} + (\mathrm{H}_{4j}\mathrm{H}_{7i}$$
$$+ \mathrm{H}_{4i}\mathrm{H}_{7j} + 2\mathrm{H}_{5i}\mathrm{H}_{8j})I^{3,1} + \mathrm{H}_{7i}\mathrm{H}_{7j}I^{4,1} + (\mathrm{H}_{3j}\mathrm{H}_{6i} + 2\mathrm{H}_{3i}\mathrm{H}_{6j})I^{0,2} + (\mathrm{H}_{4j}\mathrm{H}_{6i}$$
$$+ 2\mathrm{H}_{4i}\mathrm{H}_{6j} + \mathrm{H}_{3j}\mathrm{H}_{8i} + 2\mathrm{H}_{3i}\mathrm{H}_{8j})I^{1,2} + (2\mathrm{H}_{6j}\mathrm{H}_{7i} + \mathrm{H}_{6i}\mathrm{H}_{7j} + \mathrm{H}_{4j}\mathrm{H}_{8i} + 2\mathrm{H}_{4i}\mathrm{H}_{8j})I^{2,2}$$
$$+ (\mathrm{H}_{7j}\mathrm{H}_{8i} + 2\mathrm{H}_{7i}\mathrm{H}_{8j})I^{3,2} + 2\mathrm{H}_{6i}\mathrm{H}_{6j}I^{0,3} + 2(\mathrm{H}_{6j}\mathrm{H}_{8i} + \mathrm{H}_{6i}\mathrm{H}_{8j})I^{1,3} + 2\mathrm{H}_{8i}\mathrm{H}_{8j}I^{2,3}$$

$$F_i = \mathrm{H}_{1i}I^{0,0} + \mathrm{H}_{2i}I^{1,0} + \mathrm{H}_{3i}I^{0,1} + \mathrm{H}_{4i}I^{1,1} + \mathrm{H}_{5i}I^{2,0} + \mathrm{H}_{6i}I^{0,2} + \mathrm{H}_{7i}I^{2,1} + \mathrm{H}_{8i}I^{1,2}$$

$$
\begin{aligned}
B_{ij} = {} & \mathrm{H}_{1i}\mathrm{H}_{1j}I^{0,0} + (\mathrm{H}_{1j}\mathrm{H}_{2i} + \mathrm{H}_{1i}\mathrm{H}_{2j})I^{1,0} + (\mathrm{H}_{2i}\mathrm{H}_{2j} + \mathrm{H}_{1j}\mathrm{H}_{5i} + \mathrm{H}_{1i}\mathrm{H}_{5j})I^{2,0} + (\mathrm{H}_{2j}\mathrm{H}_{5i} \\
& + \mathrm{H}_{2i}\mathrm{H}_{5j})I^{3,0} + \mathrm{H}_{5i}\mathrm{H}_{5j}I^{4,0} + (\mathrm{H}_{1j}\mathrm{H}_{3i} + \mathrm{H}_{1i}\mathrm{H}_{3j})I^{0,1} + (\mathrm{H}_{2j}\mathrm{H}_{3i} + \mathrm{H}_{2i}\mathrm{H}_{3j} + \mathrm{H}_{1j}\mathrm{H}_{4i} \\
& + \mathrm{H}_{1i}\mathrm{H}_{4j})I^{1,1} + (\mathrm{H}_{2j}\mathrm{H}_{4i} + \mathrm{H}_{2i}\mathrm{H}_{4j} + \mathrm{H}_{3j}\mathrm{H}_{5i} + \mathrm{H}_{3i}\mathrm{H}_{5j} + \mathrm{H}_{1j}\mathrm{H}_{7i} + \mathrm{H}_{1i}\mathrm{H}_{7j})I^{2,1} \\
& + (\mathrm{H}_{4j}\mathrm{H}_{5i} + \mathrm{H}_{4i}\mathrm{H}_{5j} + \mathrm{H}_{2j}\mathrm{H}_{7i} + \mathrm{H}_{2i}\mathrm{H}_{7j})I^{3,1} + (\mathrm{H}_{5j}\mathrm{H}_{7i} + \mathrm{H}_{5i}\mathrm{H}_{7j})I^{4,1} \\
& + (\mathrm{H}_{3i}\mathrm{H}_{3j} + \mathrm{H}_{1j}\mathrm{H}_{6i} + \mathrm{H}_{1i}\mathrm{H}_{6j})I^{0,2} + (\mathrm{H}_{3j}\mathrm{H}_{4i} + \mathrm{H}_{3i}\mathrm{H}_{4j} + \mathrm{H}_{2j}\mathrm{H}_{6i} + \mathrm{H}_{2i}\mathrm{H}_{6j} + \mathrm{H}_{1j}\mathrm{H}_{8i} \\
& + \mathrm{H}_{1i}\mathrm{H}_{8j})I^{1,2} + (\mathrm{H}_{4i}\mathrm{H}_{4j} + \mathrm{H}_{5j}\mathrm{H}_{6i} + \mathrm{H}_{5i}\mathrm{H}_{6j} + \mathrm{H}_{3j}\mathrm{H}_{7i} + \mathrm{H}_{3i}\mathrm{H}_{7j} + \mathrm{H}_{2j}\mathrm{H}_{8i} + \mathrm{H}_{2i}\mathrm{H}_{8j})I^{2,2} \\
& + (\mathrm{H}_{4j}\mathrm{H}_{7i} + \mathrm{H}_{4i}\mathrm{H}_{7j} + \mathrm{H}_{5j}\mathrm{H}_{8i} + \mathrm{H}_{5i}\mathrm{H}_{8j})I^{3,2} + \mathrm{H}_{7i}\mathrm{H}_{7j}I^{4,2} + (\mathrm{H}_{3j}\mathrm{H}_{6i} + \mathrm{H}_{3i}\mathrm{H}_{6j})I^{0,3} \\
& + (\mathrm{H}_{4j}\mathrm{H}_{6i} + \mathrm{H}_{4i}\mathrm{H}_{6j} + \mathrm{H}_{3j}\mathrm{H}_{8i} + \mathrm{H}_{3i}\mathrm{H}_{8j})I^{1,3} + (\mathrm{H}_{6j}\mathrm{H}_{7i} + \mathrm{H}_{6i}\mathrm{H}_{7j} + \mathrm{H}_{4j}\mathrm{H}_{8i} + \mathrm{H}_{4i}\mathrm{H}_{8j})I^{2,3} \\
& + (\mathrm{H}_{7j}\mathrm{H}_{8i} + \mathrm{H}_{7i}\mathrm{H}_{8j})I^{3,3} + \mathrm{H}_{6i}\mathrm{H}_{6j}I^{0,4} + (\mathrm{H}_{6j}\mathrm{H}_{8i} + \mathrm{H}_{6i}\mathrm{H}_{8j})I^{1,4} + \mathrm{H}_{8i}\mathrm{H}_{8j}I^{2,4}
\end{aligned}
$$

Similarly, R^x_{ijk} and R^y_{ijk} may be expressed.

6.4.4 Element matrices for 12-noded quadrilateral elements

$$
\begin{aligned}
F_i = {} & \mathrm{H}_{1i}I^{0,0} + \mathrm{H}_{2i}I^{1,0} + \mathrm{H}_{3i}I^{0,1} + \mathrm{H}_{4i}I^{1,1} + \mathrm{H}_{5i}I^{2,0} + \mathrm{H}_{6i}I^{0,2} + \mathrm{H}_{7i}I^{2,1} + \mathrm{H}_{8i}I^{1,2} \\
& + \mathrm{H}_{9i}I^{3,0} + \mathrm{H}_{10i}I^{0,3} + \mathrm{H}_{11i}I^{3,1} + \mathrm{H}_{12i}I^{1,3}
\end{aligned}
$$

$$
\begin{aligned}
K^{xx}_{ij} = {} & \mathrm{H}_{2i}\mathrm{H}_{2j}I^{0,0} + 2(\mathrm{H}_{2j}\mathrm{H}_{5i} + \mathrm{H}_{2i}\mathrm{H}_{5j})I^{1,0} + (4\mathrm{H}_{5i}\mathrm{H}_{5j} + 3\mathrm{H}_{2j}\mathrm{H}_{9i} + 3\mathrm{H}_{2i}\mathrm{H}_{9j})I^{2,0} \\
& + 6(\mathrm{H}_{5j}\mathrm{H}_{9i} + \mathrm{H}_{5i}\mathrm{H}_{9j})I^{3,0} + 9\mathrm{H}_{9i}\mathrm{H}_{9j}I^{4,0} + (\mathrm{H}_{2j}\mathrm{H}_{4i}y + \mathrm{H}_{2i}\mathrm{H}_{4j})I^{0,1} \\
& + 2(\mathrm{H}_{4j}\mathrm{H}_{5i} + \mathrm{H}_{4i}\mathrm{H}_{5j} + \mathrm{H}_{2j}\mathrm{H}_{7i} + \mathrm{H}_{2i}\mathrm{H}_{7j})I^{1,1} + (3\mathrm{H}_{11j}\mathrm{H}_{2i} + 3\mathrm{H}_{11i}\mathrm{H}_{2j} \\
& + 4\mathrm{H}_{5j}\mathrm{H}_{7i} + 4\mathrm{H}_{5i}\mathrm{H}_{7j} + 3\mathrm{H}_{4j}\mathrm{H}_{9i} + 3\mathrm{H}_{4i}\mathrm{H}_{9j})I^{2,1} + 6(\mathrm{H}_{11j}\mathrm{H}_{5i} + \mathrm{H}_{11i}\mathrm{H}_{5j} \\
& + \mathrm{H}_{7j}\mathrm{H}_{9i} + \mathrm{H}_{7i}\mathrm{H}_{9j})I^{3,1} + 9(\mathrm{H}_{11j}\mathrm{H}_{9i} + \mathrm{H}_{11i}\mathrm{H}_{9j})I^{4,1} + (\mathrm{H}_{4i}\mathrm{H}_{4j} + \mathrm{H}_{2j}\mathrm{H}_{8i} \\
& + \mathrm{H}_{2i}\mathrm{H}_{8j})I^{0,2} + 2(\mathrm{H}_{4j}\mathrm{H}_{7i} + \mathrm{H}_{4i}\mathrm{H}_{7j} + \mathrm{H}_{5j}\mathrm{H}_{8i} + \mathrm{H}_{5i}\mathrm{H}_{8j})I^{1,2} + (3\mathrm{H}_{11j}\mathrm{H}_{4i} \\
& + 3\mathrm{H}_{11i}\mathrm{H}_{4j} + 4\mathrm{H}_{7i}\mathrm{H}_{7j} + 3\mathrm{H}_{8j}\mathrm{H}_{9i} + 3\mathrm{H}_{8i}\mathrm{H}_{9j})I^{2,2} + 6(\mathrm{H}_{11j}\mathrm{H}_{7i} + \mathrm{H}_{11i}\mathrm{H}_{7j})I^{3,2} \\
& + 9\mathrm{H}_{11i}\mathrm{H}_{11j}I^{4,2} + (\mathrm{H}_{12j}\mathrm{H}_{2i} + \mathrm{H}_{12i}\mathrm{H}_{2j} + \mathrm{H}_{4j}\mathrm{H}_{8i} + \mathrm{H}_{4i}\mathrm{H}_{8j})I^{0,3} + 2(\mathrm{H}_{12j}\mathrm{H}_{5i} \\
& + \mathrm{H}_{12i}\mathrm{H}_{5j} + \mathrm{H}_{7j}\mathrm{H}_{8i} + \mathrm{H}_{7i}\mathrm{H}_{8j})I^{1,3} + 3(\mathrm{H}_{11j}\mathrm{H}_{8i} + \mathrm{H}_{11i}\mathrm{H}_{8j} + \mathrm{H}_{12j}\mathrm{H}_{9i} \\
& + \mathrm{H}_{12i}\mathrm{H}_{9j})I^{2,3} + (\mathrm{H}_{12j}\mathrm{H}_{4i} + \mathrm{H}_{12i}\mathrm{H}_{4j} + \mathrm{H}_{8i}\mathrm{H}_{8j})I^{0,4} + 2(\mathrm{H}_{12j}\mathrm{H}_{7i} + \mathrm{H}_{12i}\mathrm{H}_{7j})I^{1,4} \\
& + 3(\mathrm{H}_{11j}\mathrm{H}_{12i} + \mathrm{H}_{11i}\mathrm{H}_{12j})I^{2,4} + (\mathrm{H}_{12j}\mathrm{H}_{8i} + \mathrm{H}_{12i}\mathrm{H}_{8j})I^{0,5} + \mathrm{H}_{12i}\mathrm{H}_{12j}I^{0,6}
\end{aligned}
$$

$$K_{ij}^{yy} = H_{3i}H_{3j}I^{0,0} + (H_{3j}H_{4i} + H_{3i}H_{4j})I^{1,0} + (H_{4i}H_{4j} + H_{3j}H_{7i} + H_{3i}H_{7j})I^{2,0}$$
$$+ (H_{11j}H_{3i} + H_{11i}H_{3j} \quad + H_{4j}H_{7i} + H_{4i}H_{7j})I^{3,0} + (H_{11j}H_{4i} + H_{11i}H_{4j}$$
$$+ H_{7i}H_{7j})I^{4,0} + (H_{11j}H_{7i} + H_{11i}H_{7j})I^{5,0} + H_{11i}H_{11j}I^{6,0} + 2(H_{3j}H_{6i}$$
$$+ 2H_{3i}H_{6j})I^{0,1} + 2(H_{4j}H_{6i} + H_{4i}H_{6j} + H_{3j}H_{8i} + H_{3i}H_{8j})I^{1,1} + 2(H_{6j}H_{7i}$$
$$+ H_{6i}H_{7j} + H_{4j}H_{8i} + H_{4i}H_{8j})I^{2,1} + 2(H_{11j}H_{6i} + H_{11i}H_{6j} + H_{7j}H_{8i} + H_{7i}H_{8j})I^{3,1}$$
$$+ 2(H_{11j}H_{8i} + H_{11i}H_{8j})I^{4,1} + (3H_{10j}H_{3i} + 3H_{10i}H_{3j} + 4H_{6i}H_{6j})I^{0,2} + (3H_{12j}H_{3i}$$
$$+ 3H_{12i}H_{3j} + 3H_{10j}H_{4i} + 3H_{10i}H_{4j} + 4H_{6j}H_{8i} + 4H_{6i}H_{8j})I^{1,2} + (3H_{12j}H_{4i}$$
$$+ 3H_{12i}H_{4j} + 3H_{10j}H_{7i} + 3H_{10i}H_{7j} + 4H_{8i}H_{8j})I^{2,2} + 3(H_{10j}H_{11i} + H_{10i}H_{11j}$$
$$+ H_{12j}H_{7i} + H_{12i}H_{7j})I^{3,2} + 3(H_{11j}H_{12i} + H_{11i}H_{12j})I^{4,2} + 6(H_{10j}H_{6i} + H_{10i}H_{6j})I^{0,3}$$
$$+ 6(H_{12j}H_{6i} + H_{12i}H_{6j} + H_{10j}H_{8i} + H_{10i}H_{8j})I^{1,3} + 6(H_{12j}H_{8i} + H_{12i}H_{8j})I^{2,3}$$
$$+ 9H_{10i}H_{10j}I^{0,4} + 9(H_{10j}H_{12i} + H_{10i}H_{12j})I^{1,4} + 9H_{12i}H_{12j}I^{2,4}$$

$$K_{ij}^{xy} = H_{2i}H_{3j}I^{0,0} + (H_{2i}H_{4j} + 2H_{3j}H_{5i})I^{1,0} + (2H_{4j}H_{5i} + H_{2i}H_{7j} + 3H_{3j}H_{9i})I^{2,0}$$
$$+ (H_{11j}H_{2i} + 2H_{5i}H_{7j} + 3H_{4j}H_{9i})I^{3,0} + (2H_{11j}H_{5i} + 3H_{7j}H_{9i})I^{4,0} + 3H_{11j}H_{9i}I^{5,0}$$
$$+ (H_{3j}H_{4i} + 2H_{2i}H_{6j})I^{0,1} + (H_{4i}H_{4j} + 4H_{5i}H_{6j} + 2H_{3j}H_{7i} + 2H_{2i}H_{8j})I^{1,1}$$
$$+ (3H_{11i}H_{3j} + 2H_{4j}H_{7i} + H_{4i}H_{7j} + 4H_{5i}H_{8j} + 6H_{6j}H_{9i})I^{2,1} + (H_{11j}H_{4i}$$
$$+ 3H_{11i}H_{4j} + 2H_{7i}H_{7j} + 6H_{8j}H_{9i})I^{3,1} + (2H_{11j}H_{7i} + 3H_{11i}H_{7j})I^{4,1} + 3H_{11i}H_{11j}I^{5,1}$$
$$+ (3H_{10j}H_{2i} + 2H_{4i}H_{6j} + H_{3j}H_{8i})I^{0,2} + (3H_{12j}H_{2i} + 6H_{10j}H_{5i} + 4H_{6j}H_{7i}$$
$$+ H_{4j}H_{8i} + 2H_{4i}H_{8j})I^{1,2} + (6H_{12j}H_{5i} + 6H_{11i}H_{6j} + H_{7j}H_{8i} + 4H_{7i}H_{8j}$$
$$+ 9H_{10j}H_{9i})I^{2,2} + (H_{11j}H_{8i} + 6H_{11i}H_{8j} + 9H_{12j}H_{9i})I^{3,2} + (H_{12i}H_{3j} + 3H_{10j}H_{4i}$$
$$+ 2H_{6j}H_{8i})I^{0,3} + (3H_{12j}H_{4i} + H_{12i}H_{4j} + 6H_{10j}H_{7i} + 2H_{8i}H_{8j})I^{1,3} + (9H_{10j}H_{11i}$$
$$+ 6H_{12j}H_{7i} + H_{12i}H_{7j})I^{2,3} + (H_{11j}H_{12i} + 9H_{11i}H_{12j})I^{3,3} + (2H_{12i}H_{6j}$$
$$+ 3H_{10j}H_{8i})I^{0,4} + (3H_{12j}H_{8i} + 2H_{12i}H_{8j})I^{1,4} + 3H_{10j}H_{12i}I^{0,5} + 3H_{12i}H_{12j}I^{1,5}$$

Similarly, B_{ij}, R_{ijk}^{x} and R_{ijk}^{y} may be expressed.

6.5 ALGORITHM TO COMPUTE ELEMENT MATRICES

1) Input for (x_i, y_i) for $i = 1, 2, 3, \cdots, NP$.
2) Formation of G matrix.
3) Computation of $H = G^{-1}$.

4) Calculation of integrals $I^{\alpha,\beta}$.
5) Computation of $K_{ij}^{xx}, K_{ij}^{yy}, K_{ij}^{xy}, \cdots, F_i$ etc. as mentioned in subsections 6.4.2 – 6.4.4.

6.6 APPLICATION EXAMPLE

To show the application of the derived formulae of this work, we consider the following two dimensional boundary value (torsion) problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2 = 0 \quad \text{within } A$$

$$u = 0 \quad \text{on } C_1^* \text{ and } \frac{\partial u}{\partial n} = 0 \quad \text{on } C_2^*$$

where C_1^* and C_2^* constitute the cross-section boundaries.

6.6.1 Finite Element Equation

The field variable u (say) governing the physical problem is

$$u = \sum_{i=1}^{NP} u_i N_i(x, y)$$

where $N_i(x, y)$ are shape functions (as given) and

$$NP = \begin{cases} 4 & \text{for the 4-noded quadrilateral} \\ 8 & \text{for the 8-noded quadrilateral} \\ 12 & \text{for the 12-noded quadrilateral} \end{cases}$$

By using the Galerkin weighted residual FE procedure, we achieve the following finite element equations,

$$[K]\{u\} = \{F\}$$

where the components of matrix $[K]$ and $\{F\}$ are

$$K_{ij} = K_{xx}^{ij} + K_{yy}^{ij} \quad \text{and} \quad F_i = 2\iint_A N_i(x, y)\,dxdy$$

5.6.2 Finite Element Procedure

The calculation process consists of the following steps

(*i*) For each element obtain components K_{ij} and F_i

(*ii*) Obtain the global FE equations for the whole system by assembling element equations.

(*iii*) Impose boundary conditions and solve for the generalized stress vectors of the whole system.

(*iv*) Calculate the torsional constant k for which $k = 2\iint_A u dx dy$

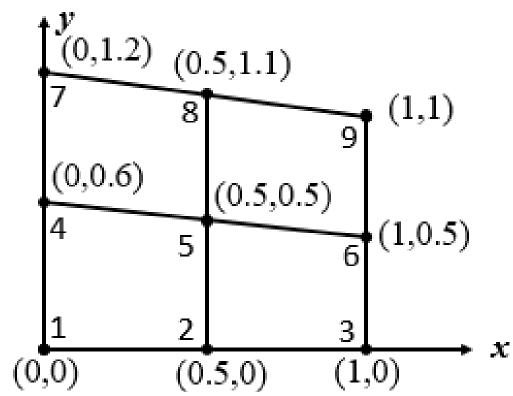
6.6.3 Test Problems

Three examples of solid cross-sections studied in (Nguyen, 1992) for which either exact or approximate and also FE solutions exists are presented. A measure of error, E_k is provided when an exact solution of the torsional constant k is available. Where

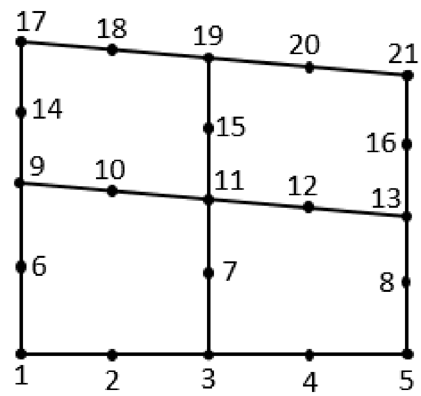
$$E_k = 100\left|1 - \frac{k}{k_{\text{exact}}}\right|$$

Example − 1: A trapezoidal cross-section

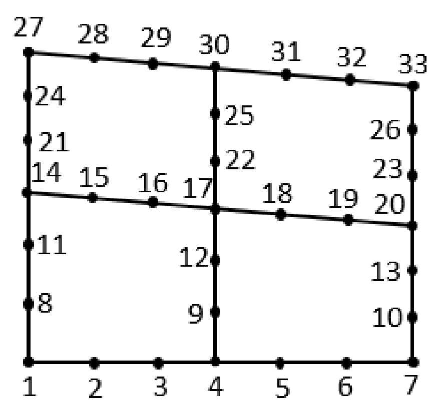
The cross-section is modeled by 4-noded, 8-noded and 12-noded quadrilaterals as shown in Fig.6.12. Computed stress function values and the torsional constant k are tabulated in Table − 6.1. Calculated value of k is in agreement with the result of (Nguyen, 1992; Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002).



(a) The FE model-1 with four 4-noded elements.



(b) The FE model-2 with four 8-noded elements.



(c) The FE model-1 with four 12-noded elements.

Fig.6.12: A trapezoidal cross-section and the FE models with four 4-noded, 8-noded and 12-noded element and global nodes.

Table – 6.1

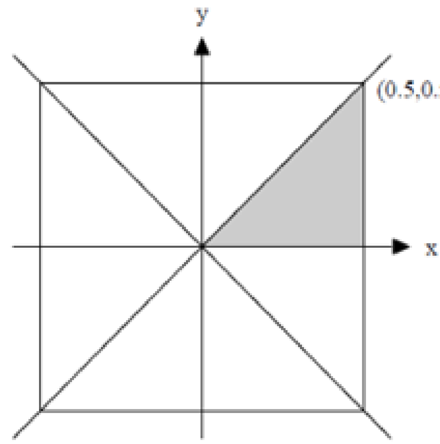
Stress function value(s), torsional constant k for example – 1.

FE Model No	U_i values				Computed Torsion Constant k
1	U_5	0.20411036036036			0.112106069137319
2	U_7	0.131915219488211	U_{12}	0.130011182721599	0.163774369530811
	U_{10}	0.128537807516107	U_{15}	0.128809621180074	
	U_{11}	0.145877253027271			
3	U_9	0.10503448562457	U_{18}	0.148355120939095	0.168497771701865
	U_{12}	0.149353737686098	U_{19}	0.103180046293444	
	U_{15}	0.101015314201091	U_{22}	0.15129104422434	
	U16	0.147387025495711	U_{25}	0.102964955448969	
	U_{17}	0.150753644844546			

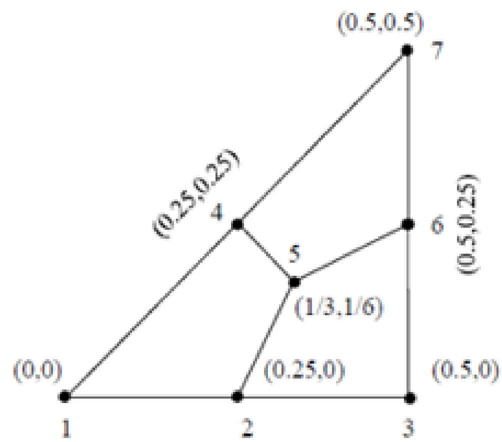
Above table shows very good convergence in the calculation of Prandtle stress function values and the torsional constant. We wish to declare that the solution for torsional constant k may be accepted as the best approximate solution.

Example – 2: A square cross- section

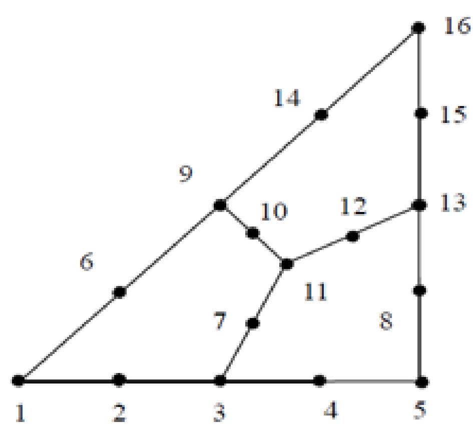
The physical geometry and FE models are shown in Fig.6.13 This cross-section has four axes of symmetry; therefore, only one-eighth of the cross-section needs to be analyzed. This fractional portion is divided into three elements (Fig.6.13(a – c)).We wish to note that at least to the knowledge of present authors the octant of a square had been modeled first time by three quadrilateral elements in (Barrett, 1999; Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000). Computed results are tabulated in Table – 6.2.



(a) Axes of symmetry of a square cross-section.



(b) The FE model-1 with three 4-noded quadrilateral elements.



(c) The FE model-1 with three 8-noded quadrilateral elements.

Fig.6.13: The octant of a square cross-section and element subdivision.

Table – 6.2

Computed stress function values, torsional constant k and error E_k for example – 2

FE Model No	U_i values				Computed Torsion Constant ***k***	E_k
1	U_1	0.15768290	U_4	0.0900712	0.1303803	7.28512%
	U_2	0.11872934	U_5	0.074753		
2	U_1	0.1471286244	U_9	0.0904859	0.1404371	0.13362%
	U_2	0.139599417	U_{10}	0.0878646		
	U_3	0.114156447	U_{11}	0.07971913		
	U_4	0.0698718347	U_{12}	0.04339872		
	U_6	0.1317489	U_{14}	0.036225977		
	U_7	0.09980412	---	--------------		
Exact torsion constant ***k*** = 0.140625						

It is to be noted that the results are in good agreement compare to (Barrett, 1999; Rathod and Sridevi, 2000) because element matrices are exactly obtained.

Example – 3: An equilateral triangular cross-section

The cross- section and FE models are shown in Fig.6.14. Due to symmetry, only one third of the original model is used. $u = 0$ is specified on sides AC and CD and $\frac{\partial u}{\partial n} = 0$ enforced at the lines of symmetry: sides AB and BD. Calculated stress function values u_i and the torsional constant k are given in Table – 6.3.

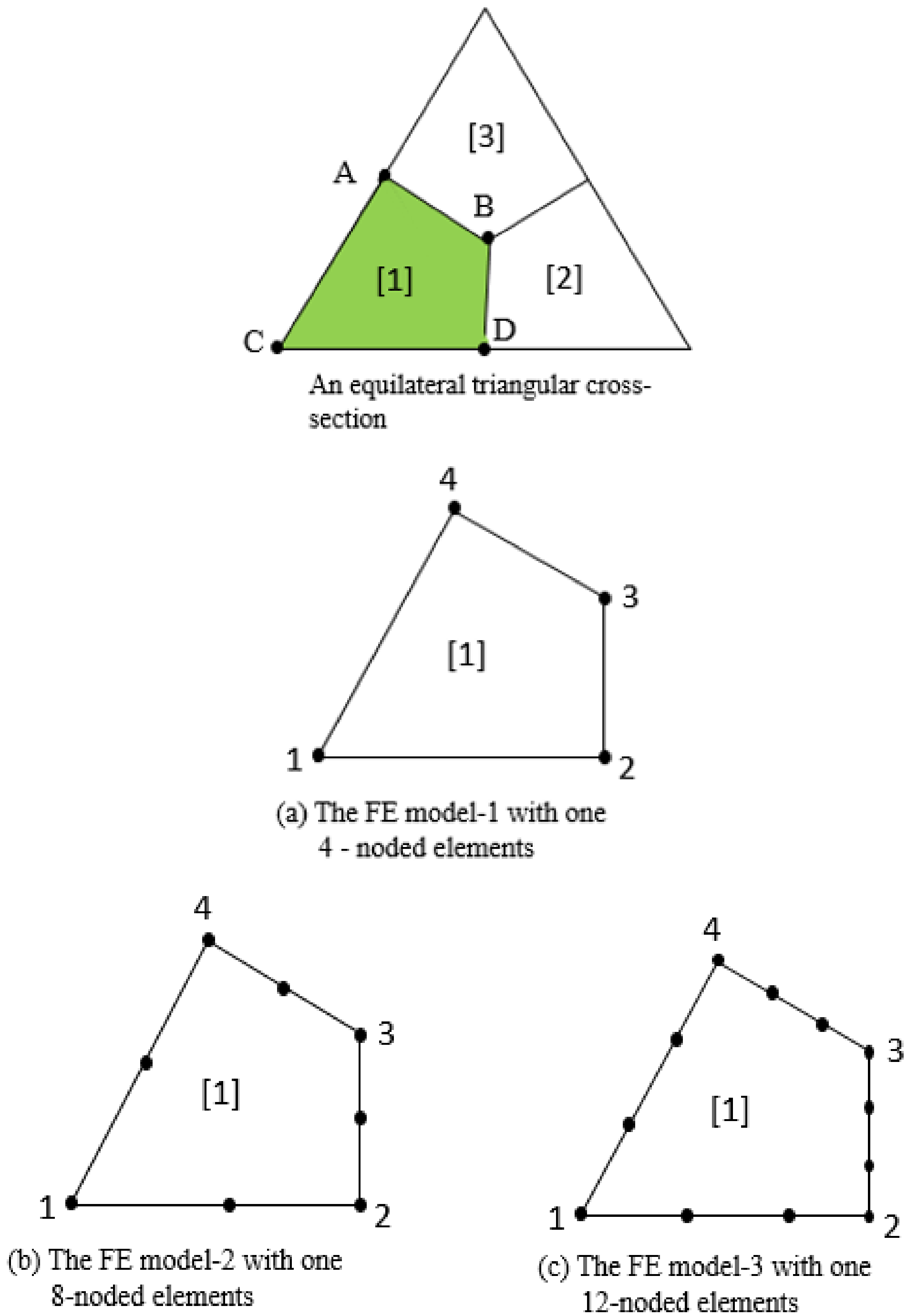


Fig.6.14: The one third of an equilateral triangular cross-section and element subdivision.

Table – 6.3

Computed stress function values, torsional constant k and error E_k for example – 3

FE Model No	U_i values				Computed Torsion Constant k	E_k
1	U_5	0.0234375	----		0.140625	549.519%
2	U_4	0.0008896	U_6	0.0240287	0.0193514660942148	10.6194%
	U_5	0.03883635	----			
3	U_5	0.03555065	U_8	0.02154	0.0220314297875825	1.7588%
	U_6	0.03914403	U_9	0.0039038		
	U_7	0.02178696	----			
Exact torsional constant is $k = 0.021650635$						

Example-4: An elliptic cross-section

The cross-section and FE models are shown previously in Fig.6.11 and Figs.6.11.2. Due to symmetry, only one-fourth of the cross-section is considered for FE models. Computed results are given in Table-6.4.

We wish to remark here that all the computed results tabulated in Tables (6.1) – (6.4) are more accurate comparing with the results reported in (Nguyen, 1992; Rathod and Islam, 1998, Karim, 2000; Rathod and Karim, 2000; Rathod and Sridevi, 2000; Rathod and Karim, 2001; Rathod and Karim, 2002).

Table – 6.4

Computed Prandtl stress values U_i and torsional constant k, for example – 4

FE Model No	U_i values				Computed Torsion Constant $\boldsymbol{k}$	E_k
1	U_1	4.9755174	U_4	4.761976	142.827008323399	5.239%
	U_2	2.8884254	U_5	1.177573		
2	U_1	3.265994	U_9	1.906576	136.882566278808	0.8589%
	U_2	3.29038	U_{10}	2.527975		
	U_3	2.871832	U_{11}	1.90699		
	U_4	1.6818382	U_{12}	1.10534		
	U_6	2.870329	U_{13}	1.0914		
	U_7	2.5464496				
Exact torsion constant $\boldsymbol{k}$ = 135.716802635079						

6.7 CONCLUSIONS

The strong mathematical foundation of Finite Element Method based on shape functions. The derivations of polynomial shape functions in local co-ordinates are comparatively easier than that of in global co-ordinates. Common practice is the use of polynomial shape functions in local co-ordinates in transformation equations. In such instances complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric/ subparametric/ superparametric elements, the presence of the determinant of the Jacobean matrix in the denominator of the stiffness matrix for which the integrands are rational functions. Precisely, to form the element stiffness matrices we are required to evaluate numerous rational integrals. Analytical integration schemes are available only for the integral of bivariate polynomial numerators with bilinear denominators. In practical situation that employs higher order elements, all the integrals are rational with the denominator of higher order bivariate expressions. Such rational integrals cannot be evaluated analytically and we are bound to employ the Gaussian quadrature schemes. Obviously, for the desired accuracy of evaluations the number of Gaussian points and weights are needed to be increased and that increases substantially the computing time. Hence, a proper balance between the accuracy and efficiency is an important task. Further, an attention is always required to select the order of the integrating rule as it is not yet totally worked out.

A suitable alternative, the use of polynomial shape functions in global coordinates in the formulation give rise integrals of polynomials which can be exactly evaluated either by the selected order of the gauss quadrature rule or by analytical schemes. In this case the main barrier is the derivation of shape functions in global coordinates for the element under considerations. Especially it is very much difficult and so cumbersome in case of the quadrilateral elements. Considering all the facts and the popularity of the quadrilateral elements, we have concentrated to derive polynomial shape functions in global coordinates and to present all the components of element matrices in bivariate polynomial form in a systematic way. So that one can use the gaussian quadrature schemes or other numerical schemes easily for exact computing all the element matrices. The technique so developed in this study is as: (1) formation of a matrix G (say) by the global nodal co-ordinates of the element geometry and then its inverse matrix H (say), and (2) the values of the integral of monomials over the element. Finally, we have employed analytical schemes and presented all the components of element matrices as the expressions of products of components of matrix H. Thus, it can be stated shortly that once the matrices G, H are formed and values of the integrals of monomials over the element are evaluated then the computation of element matrices are simply done by the product of components of matrix H only. So, it reduces many time consuming steps of FEM solution procedure and finally that reduces substantially the computational effort.

It has been clearly shown for the first time that the matrix G is singular for two types of element geometry for which bilinear shape functions (for 4-noded quadrilateral) cannot be derived but that of higher order (for 8-noded, 12-noded quadrilaterals) shape functions can be derived. Furthermore, geometrical reasons are also shown for which the matrix G is bad scaled. Through demonstrations of different type of element geometry, it has been found that such difficulties in case of derivation of shape functions may be surmounted only by changing the mesh of the domain. Hence, sincere care is needed in case of deriving polynomial shape functions in global co-ordinates for quadrilateral elements that is to ensure the matrix G is nonsingular. One can easily apply the technique to derive polynomial shape functions in local co-ordinates by forming G matrix by the local nodal co-ordinates**.** The present technique so presented to compute element matrices exactly is easy for computer coding, a computer code in MATLab® is developed which is included here with in Appendix II. The accuracy and efficiency of the formulae so presented are then demonstrated through the calculation of Prandtl stress function values and torsional constant of different type of cross sections. Comparison of computed results with the results of other researchers clearly exhibits the best accuracy of the present technique. Since explicit expressions for all types of element matrices are presented, we believe that the technique of this basic study will be contributing and attractive in the realm of application of the finite element method.

Furthermore, almost all the commercial software employs the quadrilateral elements and hence the formulae so presented in this chapter may be included to form all the element matrices exactly in efficient way.

Chapter 7

Discussion and Conclusions

Chapter 7

Discussion and conclusions

This thesis first concentrated to develop appropriate mathematical model suitable to a number of environmental management and sustainability issues like fisheries management problems and its numerical solutions. Then, it considered all the established mathematical models for all continuum mechanics problems for which a faster algorithm or a technique is the primary requirement for obtaining numerical solutions.

More specifically, considering the water resources and importance of fisheries sector of agro-based developing country Bangladesh we emphasized to develop appropriate mathematical model in order to calculate fish population. As an outcome, we finally presented the model by a system of hyperbolic partial differential equations with linear and nonlinear boundary conditions for the calculation of fish population. Further, the Thesis enhanced its intention by establishing an appropriate model for mathematical estimation of fish production performances. For obtaining numerical solutions of such models by use of Finite Volume Method, computer codes in FORTRAN compatible with the formulation are developed. Then, other codes in MATLAB are developed for analyzing and graphical presentation of computed data. Substantiation of outcomes of the developed models is then established by comparing the computed data with experimental data.

Finally, the thesis concentrated to develop the faster technique by reducing chronological steps of usual FEM solution procedure for obtaining numerical solutions of numerous continuum mechanics problems. For doing so, it stressed to present faster closed form formulae needed to form all types of element matrices for solving various problems encountered in the realm of science and engineering. For the first time it presented a technique based on the formation of a matrix by the nodal coordinates of the element under consideration for the exact computation of all types of element matrices. With this approach, the technique become faster as it now reduces many steps of usual FEM solution procedure. A computer code in MATLAB compatible with the formulation is then developed for the calculation of element matrices. Intentionally the assembly of element matrices to form global matrices is not included in code and that may be considered for developing commercial software employing higher order elements. The efficiency and accuracy of the technique is then demonstrated through application of the formulae in order to obtain the solutions of engineering problems.

Thus, the Thesis principally developed appropriate mathematical models for the calculation of fish population and population performances (size of fishes). Then it concentrated to show the applications of suitable numerical methods for obtaining best approximations of solutions of the developed models as well as existing models of field problems in an efficient manner.

The model so presented for the first time for fish population dynamics and other established models for continuum mechanics problems respectively governed by system of first order hyperbolic partial differential equations and second order (hyperbolic and elliptic) partial differential equations. We firmly believe that the developed model is applicable for all fish species for quick measurement of total fish population as well as fish sizes in length. Hence, it is expected that the model will be attracted to fisheries sectors of the globe. On the other hand, the developed technique to form exact element matrices will find immense application in the realm of science and engineering.

We devices, modifies, improves algorithms for obtaining best approximate solutions for such models by employing finite element/ volume methods. All the relevant concepts, mathematical tools, other related topics and the gradual development of the Thesis work are described in 7 (seven) chapters.

Chapter 3 is concerned with the development of a mathematical model for better management of fisheries resources. This model is nonlinear because all the parameters are chosen size dependent and age independent. The existence and uniqueness of the solution is proved. By using the finite volume method the continuous problem is discretised and then upwind explicit scheme is developed. Consequently, this model approaches the problem by the upwind explicit scheme where the consistency and stability are established. In fact, as it is investigated that the employment of the finite volume method ensures the desired accuracy of the solutions. A computer code in FORTRAN compatible with the formulations is developed for obtaining numerical solutions and other codes in MATLAB are developed to present results graphically. All the developed codes are appended for gearing up the use of the model.

In **Chapter 4**, we presented and analyzed a generic mathematical formula of a single-region size structured model which is useful for the fish production estimation. The well known and widely used von Bertalanffy's growth equation for estimation of fish size is modified and utilized here with the initial size of the fish species. The employment of the model so developed (in chapter 3) for fish population calculation utilizes initial size, birth, growth, mortality rates and the arbitrary constant of modified von Bertalanffy's growth equation as input variables. Then, the number of total fish population and the fish size at different time spans are computed. The beauty of the model for which it is interesting, it uses initial data and then accurately predicts future data like total fish population and fish sizes in length depending on time. The accuracy of the mathematical model is substantiated through the comparison of the computed and the experimental results.

Chapter 5 focused on general forms of all types of element matrices encountered in finite element solution procedure of numerous two dimensional boundary value problems**.** It is done through reviewing the governing equations and finite element formulations of two engineering problems. Finally, it has been shown that for any element (quadrilateral or triangular finite elements) there are 13 (thirteen) types integrals which are needed to evaluate in order to form such element matrices.

Chapter 6 concentrated to present close form formulae for computing all type of element matrices in an efficient way. Considering the popularity and versatility, quadrilateral finite element (linear to cubic) is considered for study. For convenience, generally shape functions in local coordinates are available for such elements. All the calculations are performed by transforming quadrilaterals into its contiguous element in local space. In such instances complications arise from two main sources, firstly the large number of integrations that need to be performed and secondly, in methods which use isoparametric/ subparametric/ superparametric elements, the presence of the determinant of the Jacobean matrix in the denominator of the stiffness matrix for which the integrands are rational functions. Precisely, to form the element stiffness matrices we are required to evaluate numerous rational integrals.

Analytical integration schemes are available only for the integral of bivariate polynomial numerators with bilinear denominators. In practical situation that employs higher order elements, all the integrals are rational with the denominator of higher order bivariate expressions. Such rational integrals cannot be evaluated analytically and so, as the simplest tool the Gaussian quadrature schemes are being used. It is reported in many researches that for the desired accuracy of evaluations of integrals of rational functions the number of Gaussian points and weights are needed to increase and that increases substantially the computing time. Hence, a proper balance between the accuracy and efficiency is an important task. Further, an attention is always required to select the order of the integrating rule as it is not yet totally worked out.

To overcome such difficulties, we intended to present in this chapter a suitable alternative by use of polynomial shape functions in global coordinates in the formulation which give rise integrals of polynomials instead of rational integrals for forming element matrices. It is well known that the faces of finite volume are finite elements and consequently the alternative technique opted for developing will be applicable in both FEM, FVM methods. But, the derivation of shape functions in global coordinates is very much difficult and so cumbersome in case of the quadrilateral elements. Considering all the facts and the popularity of the quadrilateral elements, we have concentrated to derive polynomial shape functions in global coordinates and to present all the components of element matrices in bivariate polynomial form in a systematic way. So that one can use either the gaussian quadrature schemes or other numerical schemes easily for exact computing all the element matrices.

We developed the technique, as we opted for requires only two steps: (1) formation of a matrix G (say) by the global nodal co-ordinates of the element geometry and then its inverse matrix H (say) by which first shape functions in global coordinates and then global derivatives and product of global derivatives are formed, and (2) evaluations of the integral of monomials over the element. Finally, we have employed analytical schemes and presented all the components of element matrices as the expressions of products of components of matrix H. Thus, it can be stated shortly that once the matrices G, H are formed and values of the integrals of monomials over the element are evaluated then the computation of element matrices are simply done by the product of components of matrix H only. It is now clear that the technique does not require any transformations from global to local spaces and consequently the transformation of the integrals and their numerical evaluations are not at all required. So, the employment of the developed technique reduces many steps of usual stages in Finite Element solution procedure and accordingly the computing time as well as effort reduces remarkably. On the other hand the exact integration formula to evaluate the integrals of monomials over the element is incorporated with the method for computations of element matrices. Thus, exact computation of element matrices is now possible with less computational effort and as a result proper balance between accuracy and efficiency is ensured. A suitable computer code in MATLAB compatible with the formulation is also developed. Then, like other researchers the efficiency and accuracy of the technique is demonstrated thoroughly by showing its application in order to calculate Prandtl stress function values and torsional constants of Saint-Venant torsion problems. It is now can be expected that the computer code so developed for exact computation of all types of element matrices will find frequent application in FEM, FVM solution procedure.

Further, It has been clearly shown here for the first time that the matrix G is singular for two types of element geometry for which bilinear shape functions (for 4-noded quadrilateral) cannot be derived but that of higher order (for 8-noded, 12-noded quadrilaterals) shape functions can be derived. Geometrical reasons are also shown for which the matrix G is bad scaled. Through demonstrations of different type of element geometry, it has been shown that such difficulties in case of derivation of shape functions may be surmounted only by changing the mesh of the domain. Hence, sincere care is needed in case of deriving polynomial shape functions in global co-ordinates for quadrilateral elements that is to ensure the matrix G is nonsingular. One can apply the technique for other popular finite elements like triangles.

Finally, the Thesis includes: (1) newly developed appropriate mathematical model for fish population calculation, (2) a mathematical model for calculation population sizes in length, and (3) a technique for exact computation of various types of element matrices needed in FEM, FVM solution procedures. Finite volume method is employed for obtaining numerical solutions of the developed models and accuracy is shown through the comparison of computed results of the models with the experimental results. We believe that the developed models and computer codes that employ the efficient numerical schemes for the solutions of the developed and other existing models will find immense applications in many areas of science and engineering. This Chapter 7 includes all the important conclusions relevant to the research work.

Appendix I

Computer Codes for the Formulae of the Chapter 3 and 4

The following program in FORTRAN that is Program-1 is compatible with the formulation of the developed model in chapter 3. The input is given through the file 'data' and solution of the model will be given in 'file1', 'file2' and 'filetemps'. Then the code in MATLAB that is program-2 will present solutions graphically.

For the developed model in chapter 4, first program-1 will calculate the total population and then program-3 and program-4 combinedly will calculate the length of the fish populations.

Program-1 for calculate the density of the fish population in *k* sites using FORTRAN programming

```
!=========================================================!
program population
implicit none
!=================== Declaration of Variables=================!
!-----------------------------------------------------------------------------------------!
!nl: number of subdivision of [0, L]

!nt: number of subdivision of [0, T]
!nk: number of sites
!-----------------------------------------------------------------------------------------!
!---------------------------------Integer values ----------------------------------------!
   integer :: i, nl, L, it, nt, nc, T, k, nk, n, j, i1, k1, ix, k2, k3, compt
!---------------------------------Real Values-------------------------------------------!
  real*8 :: temps, r, dt, dx,S1, S2, S3, S4, S5, sigma
```

```
real*8 :: gamma, x1, x2, x3, alpha1, alpha2, gamma1,gamma2, x01, x02, x
!--------------------------------Matrices-----------------------------------------------!
real*8, dimension(:,:), allocatable ::v, h, beta, mu, M, M1, U, Y
real*8, dimension(:), allocatable :: XX, S, P, u0max, LP
!---------------------------------Mesh-------------------------------------------------!
open (11, file='data', status='old', form='formatted')
read (11,*) nt
print*,'nt=',nt
read (11,*)T
print*, 'T=', T
read (11,*) nl
print*, 'nl=', nl
read (11,*) L
print*, 'L=', L
read (11,*) nk
print*, 'nk=', nk
read (11,*) sigma
print*, 'sigma=', sigma
read (11,*) gamma
print*, 'gamma=', gamma
read (11,*) x1
print*, 'x1=', x1
read (11,*) x2
print*, 'x2=', x2
read (11,*) x3
print*, 'x3=', x3
read (11,*) gamma1
print*, 'gamma1=', gamma1
```

```
   read (11,*) gamma2
   print*, 'gamma2=', gamma2
   read (11,*) alpha1
   print*, 'alpha1= ', alpha1
   read (11,*) alpha2
   print*, 'alpha2=', alpha2
   read (11,*) x01
   print*, 'x01=', x01
   read (11,*) x02
   print*, 'x02=', x02

   close (11)
   dx = real(L)/(nl)
    r = 0.1
    dt = r*dx
!--------------------------------Dynamic allocation--------------------------------------!
   allocate (v(1:nk, 1:nl))
   allocate (beta(1:nk, 1:nl))
   allocate (mu(1:nk, 1:nl))
   allocate (h(1:nk, 1:nl))
   allocate (M(1:nk, 1:nk))
   allocate (M1(1:nk, 1:nk))
   allocate (U(1:nk, 1:nl))
   allocate (Y(1:nk, 1:nl))
   allocate (XX(1:nl))
   allocate (S(1:nk))
   allocate (P(1:nk))
   allocate (u0max(1:nk))
```

```
   do I = 1, nl
     XX(i) = i*dx
   end do

   do i = 1, nk
     u0max(i) = 40.
   end do
!==================== Main program=======================!
!--------------------------------------Call subroutines-----------------------------------!
   call functionmu (mu, XX, nk, nl)
   call function (h, XX, sigma, gamma, nk, nl)
   call functionSpeed (v, XX, x1, nk, nl, gamma1)
   call InitialU (U, XX, nk, nl, u0max, x01, x02)
   call functionM (M, nk)
   call functionM1 (M1, nk)
   open (unit=4,file= 'file4',action="write", status="old")
   open (unit=5,file= 'file5',action="write", status="unknown")
   open (unit=1,file= 'file1',action="write", status="old")
   open (unit=2,file= 'file2',action="write", status="old")
   open (unit=9,file= 'filetemps', action="write", status="old")
!-------------------------------------------------------------------------------------------!
     temps = 0.
         nc = 0
       compt = 0
    do k1 = 1, nk
         do i = 1, nl
           write (unit= k1, fmt=*)  U(k1, i)
```

```
        end do
    end do
write (unit=9,fmt=*)temps
!--------------------------------iteration time loop---------------------------------------!
    do it  = 1, nt
           temps = temps + dt
           compt = compt + 1

       do k = 1, nk
          call somme (S, U, nl, nk)
          call totalpopulation (P, Y, nl, nk, dx)
          call functionmu (mu, XX, nk, nl)
          call function (h, XX, sigma, gamma, nk, nl)
                S(k) = S(k)*dx
                 S1=0
                S3=0

          do j = 1, nk
            S1 = S1 + M (j, k)*U (j, 1)
            S3 = S3 + M1 (j, k)
          end do

          S1 = S1 – M (k, k)*U (k, 1)
          S3 = S3 - M1 (k, k)
          Y(k,1) = (1-r*v(k,1)-dt*(mu(k,1)+h(k,1)+S3))*U(k,1) +r*S(k)+dt*S1
                        !Y(k,1)=(1-r*v(k,1)-dt*mu(k,1))*U(k,1) +r*S(k)
          do i = 2, nl
             S2=0
```

```
        S4=0
        do j = 1, nk
          S2 = S2 + M (j, k)*U (j, i)
          S4= S4 + M1 (j, k)
        end do
        S2 = S2 – M (k, k)*U (k, i)
        S4 = S4 - M1 (k, k)
       Y(k,i) = (1-r*v(k,i)-dt*(mu(k,i)+h(k,i)+S4))*U(k,i) +r*v(k,i-
                 1)*U(k,i-1) +dt*S2
    end do
 end do
if (compt ==10) then
             compt = 0
             nc = nc+1
  do k1 = 1, nk
    do i  = 1, nl
      write (unit = k1, fmt = *)  Y (k1, i)
    end do
  end do
      write (unit = 9, fmt = *) temps
 endif
call totalpopulation (P, Y, nl, nk, dx)
   U = Y
   do k1 = 1,nk
     do i = 1, nl
       U (k1,i) = Y (k1, i)
     end do
   end do
```

```
  end do

   close (unit = 1)
   close (unit = 2)
   close (unit = 3)
   close (unit = 4)
   close (unit = 5)
   close (unit = 9)
!------------------------------------ deallocation---------------------------------------!
  deallocate (v)
  deallocate (beta)
  deallocate (mu)
  deallocate (h)
  deallocate (M)
  deallocate (M1)
  deallocate (U)
  deallocate (Y)
  deallocate (XX)
  deallocate (S)
  deallocate (P)
  deallocate (u0max)
!=======================Subroutines======================!
  contains
!----------------------------------------Mortality rate----------------------------------!
   subroutine functionmu (mu, XX, nk, nl)
   implicit none
   real*8, dimension(:,:), intent(out) :: mu
   real*8, dimension(:), intent(in) :: XX
```

```
    integer :: i, k
    integer, intent(in) :: nl, nk
    do k = 1, nk
      do i = 1, nl
            mu (k,i) = 0.125 !XX(i)/(XX(i)+1)
           ! mu(2,i)=0.03
            !print*,'mu_1=',mu (k, i)
            !print*,'mu_2=', mu (2, i)
      end do
    end do
  end subroutine functionmu
!----------------------------------Harvesting policy at site k---------------------------!
   subroutine function (h, XX, sigma, gamma, nk, nl)
    implicit none
    real*8, dimension(:,:), intent(out) :: h
    real*8, dimension(:), intent(in) :: XX
    real*8, intent(in)::sigma, gamma
    integer :: i, k
    integer, intent(in) :: nl, nk

     do k = 1, nk
        do i = 1, nl
           if ((XX(i) >= gamma) .and. (XX(i) <= sigma)) then
             ! h (k, i) = 1.0/(sigma - gamma)
               h (k,i) = 0  !.02
           else
              h (k, i) = 0.
           end if
```

```
        end do
    end do
  end subroutine functionh
!------------------------------------------Grouth rate-------------------------------------!
    subroutine functionSpeed (v, XX, x1, nk, nl, gamma1)
    implicit none
    real*8, dimension(:,:), intent(out) :: v
    real*8, dimension(:), intent(in) :: XX
    integer, intent(in) :: nk, nl
    integer :: i, k
    real*8, intent(in)::x1, gamma1
    do k = 1, nk
      do i = 1, nl
            v (k, i) = 0.71!gamma1
      end do
    end do
  end subroutine functionSpeed
!-------------------------------------Initial density--------------------------------------!
    subroutine InitialU (U, XX, nk, nl, u0max, x01, x02)
    implicit none
    real*8, dimension(:,:), intent(out) :: U
    real*8, dimension(:), intent(in) :: XX, u0max
    integer :: i, k
    integer, intent(in) :: nl, nk
    real*8, intent(in)::x01, x02
    do k = 1, nk
      do i = 1, nl
          U (k, i) = u0max (k)
```

```
  end do
 end do
end subroutine InitialU
!--------------------------------Production rate of new born----------------------------!
 subroutine functionbeta (beta, XX, nk, nl, x3, x2, gamma2)
 implicit none
 real*8, dimension(:,:), intent(out) :: beta
 real*8, dimension(:), intent(in) :: XX
 integer, intent(in) :: nk, nl
 integer :: i, k
 real*8, intent(in)::x3, x2, gamma2

 do k = 1, nk
  do i = 1, nl
      beta (k, i) = 0.000245*P(k)
  end do
 end do
  end subroutine functionbeta
!----------------------------------------Migration rate------------------------------------!
 subroutine functionM (M, nk)
 implicit none
 real*8, dimension(:,:), intent(out) ::M
 integer :: i,k
 integer, intent(in) :: nk
 open (12, file = 'Mrate', status = 'old', form = 'formatted')
 do k = 1, nk
   read (12,*) (M (k, i),i = 1, nk)
 end do
```

```
    print*, 'Migration rate:'
    do k = 1, nk
      write (6,*) (M (k, i), i = 1, nk)
    end do
    close (12)
  end subroutine functionM
!--------------------------------------Emigration rate----------------------------------!
    subroutine functionM1 (M1, nk)
    implicit none
    real*8, dimension(:,:), intent(out) :: M1
    integer :: i, k
    integer, intent(in) :: nk
    open (13, file = 'M1rate', status = 'old', form = 'formatted')
    do k = 1,nk
      read (13,*) (M1 (k, i), i = 1, nk)
    end do
    print*,'Emigration rate:'
    do k = 1, nk
      write (6,*) (M1 (k, i), i = 1, nk)
    end do
    close (13)
  end subroutine functionM1
!------------------------------------------Initial data-------------------------------------!
    subroutine somme (S, U, nl, nk)
    implicit none
    real*8, dimension(:,:), intent(in):: U
    real*8, intent(out) ,dimension(:) :: S
    integer :: i, k
```

```
      integer, intent(in) :: nl, nk
      call functionbeta (beta, XX, nk, nl, x3, x2, gamma2)
      S = 0
      do k = 1, nk
        do i = 1, nl
          S(k) = S(k) + beta (k, i)*U (k, i)
        end do
      end do
    end subroutine somme
!------------------------------------------Initial data--------------------------------------!
     subroutine totalpopulation (P, Y, nl, nk, dx)
      implicit none
      real*8, dimension(:,:), intent(in):: Y
      real*8, dimension(:), intent(out) :: P
      integer :: i, k
      integer, intent(in) :: nl, nk
      real*8, intent(in):: dx
      P = 0
      do k = 1, nk
        do i = 1, nl
          P(k) =  P(k) + Y (k, i)*dx
        end do
      end do
    end subroutine totalpopulation
end program
!========================================================!
```

Program-2 for graphical representation using MATLAB programing:

```
%=====================================================%
% Program for graphical representation
clc
clear all;
nomfich = 'data.dat'
nomfich = load ('data.dat')
nt = nomfich (1)
T = nomfich (2)
nl = nomfich (3)
L = nomfich (4)
nk = nomfich (5)
sigma = nomfich (6)
gamma = nomfich (7)
x1 = nomfich (8)
x2 = nomfich (9)
x3 = nomfich (10)
gamma1 = nomfich (11)
gamma2 = nomfich (12)
alpha1 = nomfich (13)
alpha2  = nomfich (14)
x01 = nomfich (15)
x02 = nomfich (16)
dx = real(L)/nl
r = 0.5
dt = r*dx
load file2
load file1
load filetemps
```

```
nc = size (filetemps)

[m11, n11l] = size (file2)
[m11, n11l] = size(file1)

tpop1 = zeros (1, nc);
tpop2 = zeros (1, nc);

tpop1 (1) = 0.;
tpop2 (1) = 0.;
t = zeros (1, nc);
x = zeros (1, nl);
for i = 1 : nc
   t (i) = filetemps (i);
end
for j = 1 : nl
    x(j )= j * dx;
end

   for i = 1 : nc
   for j = 1 : nl
      f11 =  file2 (j + (i - 1) * (nl));
      tpop1 (i) = tpop1 (i) +  f11;
      z2(i, j) = f11;
   end
   tpop1 (i) = tpop1 (i) * dx;
end

for i = 1 : nc
   for j = 1 : nl
```

```
        f12 =  file1 (j + (i-1) *(nl));
        tpop2 (i) = tpop2 (i)  +  f12;
        z1(i, j) = f12;
    end
    tpop2 (i) = tpop2 (i) * dx;
end
nt = nt
tf = t(nc)
xfin = x(nl)
nc = nc

figure
subplot (2, 1, 1);
mesh (x, t, z1); xlabel ( 'x' );  ylabel ( 't' ); zlabel ( 'f1' );
grid on
title ( 'population 1 size-time versus' )
subplot (2, 1, 2);
mesh (x, t, z2); xlabel ( 'x' );  ylabel ( 't' ); zlabel ( 'f2' );
grid on
title ( 'population 2 size-time versus' )
figure;
subplot (2, 1, 1);
grid on
plot (t, tpop1);
title ( 'Total population 1' )
subplot (2, 1, 2);
plot (t, tpop2);
grid on
title ( 'Total population 2' )
figure;
```

```
plot ( tpop2, tpop1);
title ( 'Phase portrait pop1/pop2' )
grid on

figure;
subplot (2, 2, 1);
plot (x, z1(1,:));  xlabel ( 'x' );
title ( 'initial data pop1 , +' );
subplot (2, 2, 2);
plot (x, z2(1,:));  xlabel ( 'x' );
title ( 'initial data pop2 , +' );
subplot (2, 2, 3);
plot (t, z1(:,1));  xlabel( 't' );
title ( 'Newxborn pop1' );
subplot (2, 2, 4);
plot (t, z2(:,1));  xlabel( 't' );
title ( 'Newxborn pop2' );
%=====================================================%
```

MATLAB code in Chapter 4 as below :

Program-3 for plotting Total Fish populations

```
x = linspace ( 0, 0.5, 7)
y = load ( ' file20 ' )
plot (x, y) ;
ylim ( [0 45 ]) ;
grid on
xlabel ( ' Time Periods ' ), ylabel ( ' Length of Fish ' )
title ( ' population length-time versus ')
```

Program-4 for calculate the Length of the Fish:

```
clc
clear all
t = linspace (0, 0.5, 6) ; %0.1:0.1:0.5;
L = 15.7;   % L is initial fish size
k = 0.196;
for i = 1: 6
   L(i + 1) = L( i ) + L( i) * (1 - exp(-k * (t(i) + 0.2))) ;
end
save sizedata L
% save test.xls L -ascii
t = linspace (0, 0.5, 7) ;
x = t;
y = L
plot (x, y, '-g',' Line Width ', 2)
ylim ([0 45]) ;
grid on
xlabel ( ' Time Periods ' ),  ylabel ( ' Length of Fish ')
title ( ' population length-time versus ' )
```

Appendix II

Computer Codes for the Formulae of the Chapter 5

Following program contains six program fragments. Fragment-1 will take input of four corner nodes of quadrilateral; Fragment-2 will form the G matrix and compute inverse of G matrix i.e. H matrix; Fragment-3 will calculate all the integrals of monomials over the element; Fragment-4 will then calculate element stiffness matrix $[K]$; Fragment-5 will calculate $\{F\}$.

Code for assembly of the element matrices are not included in the program. After assembly, the Fragment-6 will solve the physical (torsion) problem.

Fragment-1 for coordinate generate:

```
%==========================================================%
function [X Y] = fem
  clc
  % For 4-node element oe = 1
  % For 8-node element oe = 2
  % For 12-node element oe = 3
  oe = 1;
  % For only one element, So nrow and ncol is equal to 1
  nrow=1; %nrow= Nomber of row
  ncol=1;     %nomber of column
%--------------Input Coordinate of the domain (See Fig. AII)-----------------------
      xt(1,1)=0;                    yt(1,1)=0;
      xt(ncol+1,1)=1.0;             yt(ncol+1,1)=0;
      xt(ncol+1,nrow+1)=1.0;        yt(ncol+1,nrow+1)=1.0;
      xt(1,nrow+1)=0;               yt(1,nrow+1)=1.2;

%-----------------------------initial Calculation-------------------------------------
      NP=oe*4;                 % NP=Number of point of each element
```

```
    NE=ncol*nrow;            % NE= Total number of element
    NGNP=(oe*ncol+1)*(nrow+1)+nrow*(ncol+1)*(oe-1);
%
%---------------Calculate Global node number of each element--------------------
% For only one element So C and R is equal to 1
jc = 0;
 C = 1; R = 1;
elno = 1;      % elno = Element number
%--------------Coordinate of corner node of each elements-------------------------
        X(elno,jc+1) = xt(C,R);
        Y(elno,jc+1) = yt(C,R);
        X(elno,jc+2) = xt(C+1,R);
        Y(elno,jc+2) = yt(C+1,R);
        X(elno,jc+3) = xt(C+1,R+1);
        Y(elno,jc+3) = yt(C+1,R+1);
        X(elno,jc+4) = xt(C,R+1);
        Y(elno,jc+4) = yt(C,R+1);
%-----------------------------------------------------------------------------------------
 if oe>1
  for I = 1 : oe-1
%-----------------------------------Side A-----------------------------------------------
          X(elno,4+I)=((oe-I)*X(elno,jc+1)+I*X(elno,jc+2))/oe;
          Y(elno,4+I)=((oe-I)*Y(elno,jc+1)+I*Y(elno,jc+2))/oe;
  end
 %----------------------------------Side B-----------------------------------------------
 for I = 1 : oe-1
          X(elno,4+I+1*(oe-1))=((oe-I)*X(elno,jc+2)+I*X(elno,jc+3))/oe;
          Y(elno,4+I+1*(oe-1))=((oe-I)*Y(elno,jc+2)+I*Y(elno,jc+3))/oe;
 end
 %----------------------------------Side C----------------------------------------------
```

```
for I = 1 : oe-1
        X(elno,4+I+2*(oe-1))=((oe-I)*X(elno,jc+3)+I*X(elno,jc+4))/oe;
        Y(elno,4+I+2*(oe-1))=((oe-I)*Y(elno,jc+3)+I*Y(elno,jc+4))/oe;
end
%----------------------------------Side D---------------------------------------------
for I = 1 : oe-1
        X(elno,4+I+3*(oe-1))=((oe-I)*X(elno,jc+4)+I*X(elno,jc+1))/oe;
        Y(elno,4+I+3*(oe-1))=((oe-I)*Y(elno,jc+4)+I*Y(elno,jc+1))/oe;
end
 %---------------------------end of discritize domain-------------------------------
end
X
Y
end
%===================End of Fragment-1================ %
```

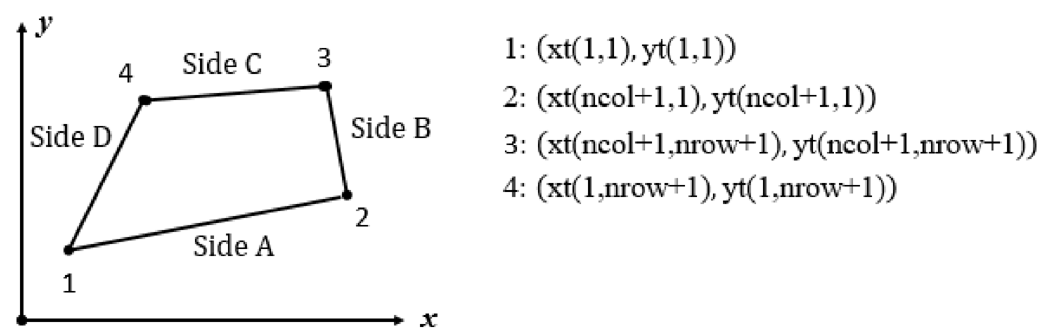


Fig. AII: Orientation of nodal co-ordinates of quadrilateral

Fragment-2 for calculate inverse of *G*:

```
%=====================================================%
function ret = calc_H()
 oe = 1 ;
[x y] = fem;      % Call Fragment-1
 if oe == 1
   for i = 1 : oe * 4
```

```
        G(i, 1) = 1;
        G(i, 2) =  x(i) ;
        G(i, 3) =  y(i) ;
        G(i, 4) =  x(i) * y(i) ;
    end
elseif oe == 2
    for i = 1 : oe * 4
        G(i, 1) = 1;
        G(i, 2) =  x(i) ;
        G(i, 3) =  y(i) ;
        G(i, 4) =  (x(i)) * (y(i)) ;
        G(i, 5) =  x(i) ^ 2 ;
        G(i, 6) =  y(i) ^ 2 ;
        G(i, 7) =  x(i) ^ 2 * y(i) ;
        G(i, 8) =  x(i) * y(i) ^ 2 ;
    end
else oe == 3
    for i = 1 : oe * 4
        G(i, 1) = 1;
        G(i, 2) =  x(i) ;
        G(i, 3) =  y(i) ;
        G(i, 4) =  x(i) * y(I );
        G(i, 5) =  x(i) ^ 2 ;
        G(i, 6) =  y(i) ^ 2 ;
        G(i, 7) =  x(i) ^ 2 * y(i) ;
        G(i, 8) =  x(i) * y(i) ^ 2;
        G(i, 9) =  x(i) ^ 3 ;
        G(i, 10) =  y(i) ^ 3 ;
        G(i, 11) =  x(i) ^ 3 * y(i) ;
        G(i, 12) =  x(i) * y(i) ^ 3 ;
```

```
   end
end
A = G       D = det (A)
H = inv(A) ;      % inverse of G
B = A * H
format long g
ret = H;
end
%====================End of Calculation H ==================%
```

Fragment- 3 for Calculation of integrals $I^{\alpha,\beta}$:

```
%==================================================%
function ret = II(h, k)
clc
n = 4; % n = Number of nodes
%h = 0; k = 0;
[x y] = fem;   % Call Fragment-1
if n == 4
   u(1) = x(1) ;       v(1) = y(1) ;
   u(2) = x(2) ;       v(2) = y(2) ;
   u(3) = x(3) ;       v(3) = y(3) ;
   u(4) = x(4) ;       v(4) = y(4) ;
   u(5) = x(1) ;       v(5) = y(1) ;
elseif n == 8
   u(1) = x(1) ;       v(1) = y(1) ;
   u(2) = x(5) ;       v(2) = y(5) ;
   u(3) = x(2) ;       v(3) = y(2) ;
   u(4) = x(6) ;       v(4) = y(6) ;
   u(5) = x(3) ;       v(5) = y(3) ;
   u(6) = x(7) ;       v(6) = y(7) ;
   u(7) = x(4) ;       v(7) = y(4) ;
```

```
    u(8) = x(8) ;        v(8) = y(8) ;
    u(9) = u(1) ;        v(9) = v(1) ;
else
    u(1) = x(1) ;        v(1) = y(1) ;
    u(2) = x(5) ;        v(2) = y(5) ;
    u(3) = x(6) ;        v(3) = y(6) ;
    u(4) = x(2) ;        v(4) = y(2) ;
    u(5) = x(7) ;        v(5) = y(7) ;
    u(6) = x(8) ;        v(6) = y(8) ;
    u(7) = x(3) ;        v(7) = y(3) ;
    u(8) = x(9) ;        v(8) = y(9) ;
    u(9) = x(10) ;       v(9) = y(10) ;
    u(10) = x(4) ;       v(10) = y(4) ;
    u(11) = x(11) ;      v(11) = y(11) ;
    u(12) = x(12) ;      v(12) = y(12) ;
    u(13) = u(1) ;       v(13) = v(1) ;
end
u
v
ret = 0;
for I = 1 : n
    for p = 0 : h + 1
        for q = 0 : k
            const = (nchoosek(h + 1, p) * nchoosek(k, q)) / (p + q + 1) ;
            ret = ret + const * (u(i) ^ (h + 1 - p)) * ((u(i + 1) - u(i)) ^ p) * (v(i) ^
                (k - q)) *((v(i + 1) - v(i)) ^ (q + 1));
             a = ret / (h + 1);
        end
    end
end
```

```
format long g
ret = a ;
end

%====================End of Calculation I ==================%
```

Fragment-4 for calculate the components of matrix $[K]$:

```
%=======================================================%
function ret = calc_K()
tic
oe = 1;
H = calc_H();        % Call Fragment-2
if oe == 1
  for i = 1 : oe * 4
    for j = 1 : oe * 4
      Kxx(i,j) = H(2,i)*H(2,j)*II(0,0)+(H(2,j)*H(4,i)+H(2,i)*H(4,j))*II(0,1)
             +H(4,i)*H(4,j)*II(0,2);

      Kyy(i,j) = H(3,i)*H(3,j)*II(0,0)+(H(3,j)*H(4,i)+H(3,i)*H(4,j))*II(1,0)
             +H(4,i)*H(4,j)*II(2,0);
    end
  end
elseif oe == 2
  for i = 1 : oe * 4
    for j = 1 : oe * 4
      Kxx(i,j) = H(2,i)*H(2,j)*II(0,0)+2*(H(2,j)*H(5,i)+H(2,i)*H(5,j))*
              II(1,0)+4*H(5,i)*H(5,j)*II(2,0)+(H(2,j)*H(4,i)+H(2,i)*
              H(4,j))*II(0,1) +2*(H(4,j)*H(5,i)+H(4,i)*H(5,j)+H(2,j)*
              H(7,i)+H(2,i)*H(7,j))*II(1,1)+4*(H(5,j)*H(7,i)+H(5,i)*
              H(7,j))*II(2,1)+(H(4,i)*H(4,j)+   H(2,j)*H(8,i)+H(2,i)*
              H(8,j))*II(0,2)+2*(H(4,j)*H(7,i)+H(4,i)*H(7,j)+ H(5,j)*
```

```
                    H(8,i)+H(5,i)*H(8,j))*II(1,2)+4*H(7,i)*H(7,j)*II(2,2)+
                    (H(4,j)*H(8,i)+H(4,i)*H(8,j))*II(0,3)+2*(H(7,j)*
                    H(8,i)+H(7,i)* H(8,j)) *II(1,3) +H(8,i)*H(8,j)*II(0,4);

        Kyy(i,j) = H(3,i)*H(3,j)*II(0,0)+(H(3,j)*H(4,i)+H(3,i)*H(4,j))*II(1,0)
                   +(H(4,i)*H(4,j)+H(3,j)*H(7,i)+H(3,i)*H(7,j))*II(2,0)+
                    (H(4,j)*H(7,i) +H(4,i)*H(7,j))*II(3,0)+H(7,i)*H(7,j)*
                    II(4,0)+2*(H(3,j)*H(6,i) +H(3,i)*H(6,j))*II(0,1)+2*(H(4,j)*
                    H(6,i)+H(4,i)*H(6,j) +H(3,j)*H(8,i)+H(3,i)*H(8,j))*II(1,1)
                  +2*(H(6,j)*H(7,i)+H(6,i)*H(7,j)+H(4,j)*H(8,i)+H(4,i)*
                   H(8,j))*II(2,1)+2*(H(7,j)*H(8,i)+H(7,i)* H(8,j))*II(3,1)+
                   4*H(6,i)*H(6,j)*II(0,2)+4*(H(6,j)*H(8,i) +H(6,i)*
                   H(8,j))*II(1,2)+ 4*H(8,i)*H(8,j)*II(2,2);
    end
  end
else
  for i=1:oe*4
    for j=1:oe*4
        Kxx(i,j) = H(2,i)*H(2,j)*II(0,0)+2*(H(2,j)*H(5,i)+H(2,i)*H(5,j))*
                    II(1,0) +(4*H(5,i)*H(5,j)+3*H(2,j)*H(9,i)+3*H(2,i)*
                 H(9,j))*II(2,0)+. 6*(H(5,j)*H(9,i)+H(5,i)*H(9,j))*II(3,0)+
                 9*H(9,i)*H(9,j)*II(4,0) +(H(2,j)*H(4,i)+H(2,i)*H(4,j))*II(0,1)+
                 2*(H(4,j)*H(5,i)+H(4,i)*H(5,j) + H(2,j)*H(7,i)+H(2,i)*H(7,j))*
                 II(1,1)+(3*H(11,j)*H(2,i)+3* H(11,i)* H(2,j)+4*H(5,j)*H(7,i)+
                 4*H(5,i)*H(7,j)+3*H(4,j)*H(9,i)+3*H(4,i)* H(9,j))*II(2,1)+6*
                 (H(11,j)*H(5,i)+H(11,i)*H(5,j)+H(7,j)*H(9,i) +H(7,i)*H(9,j))*
                 II(3,1)+9*(H(11,j)*H(9,i)+H(11,i)*H(9,j))*II(4,1)+ H(2,j)*
                 H(8,i)+H(2,i)*H(8,j))*II(0,2)+2*(H(4,j)*H(7,i) +H(4,i)*H(7,j)+
                 H(5,j)*H(8,i)+H(5,i)*H(8,j))*II(1,2)+ (3*H(11,j)*H(4,i) +3*
                 H(11,i)*H(4,j)+4*H(7,i)*H(7,j)+3*H(8,j)*H(9,i)+3*H(8,i)*
```

H(9,j))*II(2,2)+6*(H(11,j)*H(7,i)+H(11,i)*H(7,j))*II(3,2)+
9*H(11,i)*H(11,j)*II(4,2) +(H(12,j)*H(2,i)+H(12,i)*H(2,j)+
H(4,j)*H(8,i)+H(4,i)*H(8,j))* II(0,3)+2*(H(12,j)*H(5,i)+
H(12,i)*H(5,j)+ H(7,j)*H(8,i)+H(7,i)* H(8,j))*II(1,3)+3*
(H(11,j)*H(8,i)+H(11,i)*H(8,j)+H(12,j)*H(9,i) +H(12,i)*
H(9,j))*II(2,3)+(H(12,j)*H(4,i)+ H(12,i)*H(4,j)+H(8,i)*
H(8,j))*II(0,4)+2*(H(12,j)*H(7,i)+H(12,i)*H(7,j))*II(1,4)+
3*(H(11,j)*H(12,i)+H(11,i)*H(12,j))*II(2,4)+(H(12,j)*H(8,i)+
H(12,i)*H(8,j))*II(0,5)+H(12,i)*H(12,j)*II(0,6);

Kyy(i,j) = H(3,i)*H(3,j)*II(0,0)+(H(3,j)*H(4,i)+H(3,i)*H(4,j))*II(1,0)
+(H(4,i)*H(4,j)+H(3,j)*H(7,i)+H(3,i)*H(7,j))*II(2,0)+
(H(11,j)*H(3,i) +H(11,i)*H(3,j)+H(4,j)*H(7,i)+H(4,i)*
H(7,j))*II(3,0)+(H(11,j)*H(4,i) +H(11,i)*H(4,j)+H(7,i)*
H(7,j))*II(4,0)+(H(11,j)*H(7,i)+ H(11,i)*H(7,j))*II(5,0)+
H(11,i)*H(11,j)*II(6,0)+2*(H(3,j)*H(6,i)+H(3,i)*
H(6,j))*II(0,1)+2*(H(4,j)*H(6,i)+H(4,i)*H(6,j)+H(3,j)*H(8,i)
+H(3,i)*H(8,j))*II(1,1)+2*(H(6,j)*H(7,i)+H(6,i)*H(7,j)+
H(4,j)*H(8,i) +H(4,i)*H(8,j))*II(2,1)+2*(H(11,j)*H(6,i)+
H(11,i)*H(6,j)+H(7,j)* H(8,i)+H(7,i)*H(8,j))*II(3,1)+2*
(H(11,j)*H(8,i)+H(11,i)*H(8,j))* II(4,1)+(3*H(10,j)*H(3,i)+
3*H(10,i)*H(3,j)+4*H(6,i)*H(6,j))*II(0,2) +(3*H(12,j)*
H(3,i)+3*H(12,i)*H(3,j)+3*H(10,j)*H(4,i)+3*H(10,i)*
H(4,j)+4*H(6,j)*H(8,i)+4*H(6,i)*H(8,j))*II(1,2)+
(3*H(12,j)*H(4,i) +3*H(12,i)*H(4,j)+3*H(10,j)*H(7,i)+3*
H(10,i)*H(7,j)+4*H(8,i)* H(8,j))*II(2,2)+3*(H(10,j)*H(11,i)+
H(10,i)*H(11,j)+ H(12,j)*H(7,i) +H(12,i)*H(7,j))*II(3,2)+
3*(H(11,j)*H(12,i)+H(11,i)*H(12,j))*II(4,2) +6*(H(10,j)*
H(6,i)+H(10,i)*H(6,j))*II(0,3)+ 6*(H(12,j)*H(6,i) +H(12,i)*
H(6,j)+H(10,j)*H(8,i)+H(10,i)*H(8,j))*II(1,3)+6*(H(12,j)*

```
                H(8,i)+H(12,i)*H(8,j))*II(2,3)+ 9*H(10,i)*H(10,j)*II(0,4)+ 9*
               (H(10,j)*H(12,i)+H(10,i)*H(12,j))*II(1,4)+9*H(12,i)*H(12,j)*
                II(2,4);
      end
   end
end
%format long g
K = Kxx + Kyy;
ret = K;
toc
end

%====================End of Calculation K =================%
```

Fragment-5 for calculate *F* :

```
%=====================================================%
function ret = calc_F()
tic
clc
oe = 2;
H = calc_H();          % Call Fragment-2
if oe==1
   for i=1:oe*4
      F(i) = H(1,i)*II(0,0)+H(2,i)*II(1,0)+H(3,i)*II(0,1)+H(4,i)*II(1,1);
   end
elseif oe==2
   for i=1:oe*4
      F(i) =H(1,i)*II(0,0)+H(2,i)*II(1,0)+H(3,i)*II(0,1)+H(4,i)*II(1,1)
             +H(5,i)*II(2,0)+H(6,i)*II(0,2)+H(7,i)*II(2,1)+H(8,i)*II(1,2);
   end
else
```

```
  for i=1:oe*4
     F(i) = H(1,i)*II(0,0)+H(2,i)*II(1,0)+H(3,i)*II(0,1)+H(4,i)*II(1,1)+
            H(5,i)*II(2,0) +H(6,i)*II(0,2)+H(7,i)*II(2,1)+H(8,i)*II(1,2)+
            H(9,i)*II(3,0)+H(10,i)* II(0,3)+H(11,i)*II(3,1)+H(12,i)*II(1,3);
  end
end
 %format long g
ret = F;
toc
end
%====================End of Calculation F =================%
```

Fragment- 6 for calculate *Stress Functions* and *Torsion constant*:

```
%=======================================================%
function ret = calc_u()
clc
k = [ ]   % input the [K] after assembly
f = [ ]'   % input the {F} after assembly
d = det(k);
k1 = inv(k);
%format short
A = k * k1;
format long g
u = 2 * k1 * f
t = 2 * u' * f
ret = t;
 end
%====================End of Calculation  ==================%
```

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List of Publications

1. ***Razwan Ahamad*** *and M.S. Karim,* "**Study on derivation of shape functions in global coordinates and exact computation of element matrices for quadrilateral finite elements",** IOSR Journal of Mathematics (IORS-JM), Vol. 12, Issue 6, Nov. –Dec. 2016, page 90–103.(Included in Chapter 6)
2. ***Razwan Ahamad****, M.S. Karim,.and M.A.H. Mithu,* " **Mathematical Estimation of Production Performance of Fish Population**", Int. Advanced Research Journal in Science, Engineering and Technology, Vol. 3, Issue 8, August 2016, page 144–149. (Included in Chapter 4)
3. ***Razwan Ahamad****, M.S. Karim,.and M.A.H. Mithu,* "**Development of a mathematical model for better management of fisheries resources**", Int. J. Mathematical Modelling and Numerical Optimisation, Vol. 6, No. 3, 2015, page 198–222. (Included in Chapter 3)