A Study of Recognition and Enumeration for Decomposable Ordered Sets

A thesis for the degree of
Doctor of Philosophy in Mathematics

by

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Statement of Authorship

I do hereby declare that the results included in this thesis have been obtained by me under the supervision of Prof. Dr. Md. Rashed Talukder and Prof. Dr. Shamsun Naher Begum in the Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh. No other person's work has been used without due acknowledgment in the main text of the thesis. I also declare that this thesis has not been submitted elsewhere for other purposes except for publications.

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Abstract

In this research, we consider mainly two repeatedly studied combinatorial problems in the theory of posets (partially ordered sets). The first one, known as the Recognition Problem, is to recognize those classes of posets that satisfy some common structural properties. Here, we give the recognitions of the most computationally tractable classes of posets, namely the decomposable posets. Well-known classes of the decomposable posets are the classes of P-graphs, P-series, and series-parallel posets. Also, we introduce the notions of the classes of factorable posets and composite posets. Due to many computational aspects of the incidence matrices, they have classical applications in recognizing various classes of posets and graphs. For this, we introduce the notion of poset matrix, an incidence matrix, to represent finite posets. Here, we define the order relation in a square (0,1)-matrix and give an association of this matrix to the posets. We show that every poset matrix can be relabeled to an upper (equivalently, lower) triangular matrix that represents a unique poset up to isomorphism. We introduce the notions of the ordinal sum, ordinal product, and composition of matrices. We establish the algebraic interpretations of the direct sum, ordinal sum, Kronecker product, ordinal product, and composition of matrices in the case of poset matrices. We give the matrix recognitions of the classes of P-graphs, P-series, series-parallel posets, factorable posets, and composite posets by using the poset matrix. Finally, we give a matrix recognition of the class of all decomposable posets that generalizes most of the above results regarding the matrix recognitions of posets.

The second problem, known as the Enumeration Problem, is to count the number of pairwise nonisomorphic posets with a certain number of elements belonging to a particular class of posets. Among several methods for the enumeration of posets, here, we consider the exact enumeration method. The algorithmic methods for the enumerations of some classes of decomposable posets considered in the previous researches are of type generate-one and count-one. As a result, the running time of these algorithms grows more rapidly even though the posets are significantly small in size. It is mainly due to the recursions in generating pairwise nonisomorphic posets that make these algorithms highly time-complex. Therefore, it was always a great challenge to give polynomial-time algorithms to make some enumeration process time-efficient. Since the generating methods for the decomposable posets seem to consist of the recursive constructions, algorithmic methods for the enumerations of such posets were ignored by some authors. Here, we give an exact enumeration method for the unlabeled P-series and series-parallel posets by using the poset matrix. In both cases, we give the enumeration of the unlabeled disconnected posets according to the number of connected direct terms of the posets. In the case of the unlabeled connected series-parallel posets, we give the enumeration of the posets according to the number of ordinal terms that are either the singleton or disconnected posets. For these, we use the results regarding the matrix recognitions of the classes of P-series and series-parallel posets. We also give some algorithms to determine the parameters involved in the enumeration formulae, and finally, compute the number of unlabeled posets. We show that these enumeration algorithms have polynomial-time complexities. Moreover, we implement these enumeration algorithms into the computer and obtain the numbers of unlabeled P-series up to 75 elements and the numbers of unlabeled series-parallel posets up to 33 elements.

Contents

CHAPTER 1

Introduction

Due to the computational tractability property, some classes of posets (partially ordered sets) provide the basic structures of several applied and theoretical problems in various fields of science and engineering [**40**]. As a result, in many fields of applied mathematics and information science, posets are considered as fundamental structures for visualizing and analyzing information data [**4**]. Such a procedure for visualizing and analyzing a data set becomes the most efficient one when the data in the set can be organized by linear ordering. On the other hand, these procedures take more complex forms when only some partial orderings about these data can be known. According to the results obtained by Butler [**8**], Chaunier and Lygeros [**10**], and Heitzig-Reinhold [**25**], we observe that a data set may have one of the possible structures of 2,567,284 unlabeled posets with only 10 elements and 1,338,193,159,771 unlabeled posets with only 14 elements. Later on, Brinkmann and Mckay [**7**] computed the number of unlabeled posets up to 16 elements by extending the method developed by Heitzig and Reinhold [**25**] and introduced by Butler [**8**]. They showed that the number of 16-element unlabeled posets equals the 16-digit number 4,483,130,665,195,087 which is greater than 3000 times of the number of 14-element unlabeled posets. They obtained this number as the sum of more than 100 big numbers. Since the number of finite posets increases exponentially with the number of elements in the posets, it becomes an important issue to observe their structural properties. We find that one of the frequently studied classes of computationally tractable posets is the class of decomposable posets. To maximize the efficiency, methods for solving many optimization problems on structure theory begin with some decomposition techniques. These techniques are used to reduce a bigger structure into smaller ones of the same kind, like posets into autonomous sets [**27, 31**], graphs into clumps [**5**], comparability graphs into stable sets [**50**], schedules into job-modules [**36**], networks into simplifiable subnetworks [**51**], and so on. Therefore, different methods for the recognitions and enumerations of various classes of decomposable posets and graphs were considered by numerous authors. These intuitions motivate us to consider the following two repeatedly studied combinatorial problems regarding the classes of decomposable posets.

(1) **The Recognition Problem:** To recognize the classes of posets that satisfy some common structural properties.
(2) **The Enumeration Problem:** To count the total number of pairwise nonisomorphic posets belonging to a particular class of posets.

Firstly, we consider the Recognition Problem. Among the classes of decomposable posets, we recall the classes of P-graphs, P-series, and series-parallel posets. In addition, we introduce the notions of the classes of factorable posets and composite posets. We give the matrix recognitions of the classes of P-graphs, P-series, series-parallel posets, factorable posets, and composite posets. We also give a matrix recognition of the class of all decomposable posets that generalizes most of the above results regarding the matrix recognition of posets. In general, various classes of posets and graphs are recognized by using the methods of set theoretic concepts, forbidden configurations, incidence matrices, and algorithms. Due to many computational aspects of incidence matrices, they have classical applications in the recognitions of various classes of posets [**31, 46, 48, 52**]. Therefore, for the recognitions of the aforementioned classes of posets, we consider the method of incidence matrices. For this, we introduce the notion of poset matrix, an incidence matrix, to represent finite posets. Here, we define the order relation in a square (0,1)-matrix and give its association to the posets. Various operations on matrices were considered in literature due to their classical applications in many fields of science and engineering. Among these we recall the direct sum and Kronecker product of matrices. Also, we introduce the notions of the ordinal sum, the ordinal product, and composition of matrices. Then we describe the algebraic interpretations of these operations in the case of poset matrices. Briefly, we do the following.

(1) We show that the matrix transpose of a poset matrix represents the poset dual to the poset represented by the poset matrix.
(2) We describe the interpretations of relabeling (simultaneous interchanges of the rows and columns) of a poset matrix to the posets.
(3) We show that every poset matrix can be relabeled into an upper (equivalently, lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling.
(4) We give the algebraic interpretations of the direct sum, ordinal sum, Kronecker product, ordinal product, and composition of matrices in the case of poset matrices.
(5) We define the properties of block of 0s, block of 1s, and complete blocks of 1s on a poset matrix. Then we give the matrix recognitions of the classes of P-graphs, P-series, and series-parallel posets.
(6) We also define the properties of transitive blocks of 1s and transitive blocks of poset matrices on a block poset matrix. Then we give the matrix recognitions of the classes of factorable posets and composite posets.
(7) Finally, we give a matrix recognition of the class of all decomposable posets and show that it generalizes most of the above results regarding the matrix recognition of posets.

Secondly, we consider the Enumeration Problem. The methods used repeatedly for the enumerations of various classes of posets are the exact enumeration, asymptotic enumeration, enumeration by generating functions, and algorithmic enumeration. Among these, for the enumerations of some classes of the decomposable posets, we consider mainly the exact enumeration method. For well-known algorithmic enumerations, we refer the readers to [**2, 8, 10, 12, 25**]. The algorithms for the enumerations of some classes of posets considered in these previous researches are merely of type generate-one and count-one. Therefore, the running times of these algorithms grow more rapidly even though the posets are significantly small in size. It is mainly due to the recursive process for generating some specific types of posets. This recursion is one of the major things that make the enumeration algorithms so time-complex. Therefore, it was always a great challenge to give polynomial-time algorithms to make certain enumeration process time-efficient. Since the generating methods for the P-series and series-parallel posets seem to consist of recursions, algorithmic methods for the enumeration of these classes of posets are ignored by some researchers [**15**]. In this thesis, firstly, we give an exact enumeration of the unlabeled disconnected posets belonging to a class that is closed under the direct sum. This gives, consequently, some exact enumerations of the classes of unlabeled P-series and series-parallel posets. Here, we give the enumeration of the unlabeled disconnected posets according to the number of connected direct terms of the posets. In the case of the unlabeled connected series-parallel posets, we give the enumeration of the posets according to the number of ordinal terms that are either the singleton or disconnected posets. Also, we give some algorithms with polynomial time-complexities to determine the parameters involved in the enumeration formulae as well as to compute the total number of unlabeled posets. Briefly, we do the following.

(1) We give the recognitions of the connected and disconnected posets by using the poset matrix. Then we give the matrix recognitions of the classes of connected and disconnected P-series and series-parallel posets.
(2) We give an exact enumeration of the unlabeled disconnected posets belonging to a class that is closed under the direct sum. This result establishes that the enumeration of unlabeled posets belonging to a class of posets that is closed under the direct sum depends mainly on the enumeration of the connected posets belonging to the class.
(3) We give the exact enumerations of the unlabeled disconnected P-series and series-parallel posets by using the above result regarding the exact enumeration of unlabeled disconnected posets. In both cases, we give the enumeration of the unlabeled disconnected posets according to the number of connected direct terms of the posets.

(4) We give an exact enumeration of the unlabeled connected P-series by using the poset matrix. We show that the number of unlabeled connected P-series (equivalently, unlabeled nontrivial P-graphs) can be given by an explicit formula.

(5) We also give an exact enumeration of the unlabeled connected series-parallel posets. Here, we give the enumeration of the unlabeled connected posets according to the number of ordinal terms that are either the singleton or disconnected posets.

(6) Also, we give some algorithms to determine the parameters involved in the enumeration formulae and to find, ultimately, the number of unlabeled P-series and series-parallel posets with a certain number of elements. We show that all the enumeration algorithms run in polynomial times.

(7) Finally, we implement the enumeration algorithms into the computer and obtain some numerical results regarding the number of unlabeled P-series up to 75 elements and the number of unlabeled series-parallel posets up to 33 elements.

The body of the thesis comprises four consecutive chapters excluding the introduction, Chapter 1. The first of these, Chapter 2, consists of the basic terminologies related to the posets and their incidence structures. Here, we include the explanatory examples, descriptive diagrams, important remarks, and useful observations regarding the constructions of different types of posets that will be recalled throughout this thesis. We also include brief discussions on the common methods for the recognitions and enumerations of posets and related mathematical structures.

In Chapter 3, we give the foundation of the methods for the solutions to the problems considered in this research. We introduce the notion of the poset matrix to represent finite posets and give its association to the posets. We study the interpretations of different forms of the poset matrices. We give the interpretations regarding the relabeling, the interchanges of rows and columns simultaneously, in a poset matrix, and the matrix transpose of a poset matrix. We recall the notions of the direct sum and the frequently studied Kronecker product of matrices. We introduce the notion of ordinal sum, ordinal product, and a composition of matrices. We give the interpretations of these sums, products, and the composition in the case of poset matrices.

In Chapter 4, we give some solutions to the problem of recognition of the classes of decomposable posets. We define the properties of block of 0s, block of 1s, and complete blocks of 1s on a poset matrix. Then we give the matrix recognitions of the classes of P-graphs, P-series, and series-parallel posets by using the poset matrix. We also define the property of the transitive blocks of 1s and the transitive blocks of poset matrices on a block poset matrix. Then we give the matrix recognitions of the classes of factorable posets and composite posets. Finally, we give a matrix recognition of the class of all decomposable posets and show that it generalizes most of the above results regarding the matrix recognition of posets.

In Chapter 5, we give the solutions to the problem of enumeration of two well-known classes of decomposable posets. Firstly, we recall the results regarding the matrix recognition of posets and give the recognition of the connected and disconnected P-series and series-parallel posets by using the poset matrix. Then we give exact enumerations of the unlabeled disconnected posets belonging to a class of posets that is closed under the direct sum of posets. Then we show that the aforesaid enumeration method gives an exact enumeration of the class of P-series and the class of series-parallel posets. We show that the enumeration algorithms run in polynomial times. We include the numerical results for the n-element unlabeled P-series for $1 \leq n \leq 75$. Also, we include the numerical results for the n-element unlabeled series-parallel posets for $1 \leq n \leq 33$.

In addition, as an appendix of the thesis, we give the pseudocodes developed for the implementations of enumeration algorithms into the computer and the details of the numerical results obtained throughout the process.

CHAPTER 2

Posets and Basic Terminologies

In this chapter, we give the essential definitions, explanatory examples, descriptive diagrams, notable remarks, and useful observations that constitute the foundation of this thesis and helpful to start with. For further details on the basics of posets, we would like to refer the readers to the classical books by Davey and Priestley [**14**] and Grätzer [**21**].

In many fields of applied mathematics and information science, posets are considered as fundamental structures for visualizing and analyzing information data [**4**]. The procedure for visualizing and analyzing a data set become the most efficient one when the data in the set can be organized by the linear ordering. On the other hand, these methods take more sophisticated forms when only some partial orderings about these data can be known. As a result, it becomes an important issue to observe the structural properties of various classes of posets. These motivate us to consider some combinatorial problems regarding the recognition and enumeration of the classes of decomposable posets in this thesis.

In Section 2.1, we recall some basic terminologies related to the posets and their incidence structures related to various compositions of posets.

In Section 2.2, we shortly recall some special kinds of posets that are considered repeatedly in the literature and have strong connections mainly to the recognition and enumeration of posets.

In Section 2.3, we recall the basic operations related to the sum, product, and composition of posets. With these operations in posets, we obtain new posets by using the old posets that induce some classes of decomposable posets.

In Section 2.4, we recall the common subclasses of decomposable posets such as the classes of P-graphs, P-series, and series-parallel posets. Here, we also introduce the notions of the classes of factorable posets and composite posets.

In Section 2.5, we include a quick review on some common methods for the recognition of several classes of posets, and in Section 2.6, we include a brief discussions on the repeatedly studied methods for the enumeration of posets.

2.1. Useful definitions

2.1.1. Poset.

Definition 2.1.1 A *poset* (*partially ordered set*) is a structure $\mathbf{P} = \langle P, \leqslant \rangle$ consisting of the nonempty set P with the order $\leqslant$ on P, that is, the relation $\leqslant$ is reflexive, antisymmetric, and transitive.
Here the set P is called the *underlying set* or *ground set* of the poset $\mathbf{P}$. The poset $\mathbf{P}$ is called *finite* if the underlying set P is finite.

Examples 2.1.1

(1) $\langle P(X), \subseteq \rangle$, where $P(X)$ is the power set of the nonempty set X and $\subseteq$, the relation of *set inclusion*, is an order on $P(X)$. In particular, for $X = \{x, y\}$, we have $P(X) = \{\emptyset, \{x\}, \{y\}, X\}$ and $\subseteq = \{(\emptyset, \emptyset), (\emptyset, \{x\}), (\emptyset, \{y\}), (\emptyset, X), (\{x\}, \{x\}), (\{y\}, \{y\}), (\{x\}, X), (\{y\}, X), (X, X)\}$.

(2) $\langle D(n), | \rangle$, where $D(n)$ is the set of all divisors of the natural number n and $|$, the relation of *divisibility*, is an order on $D(n)$. Particularly, for $n = 12$, we have $D(12) = \{1, 2, 3, 4, 6, 12\}$ and $| = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (12, 12)\}$.

(3) $\langle \mathbb{N}, \leq \rangle$, where $\mathbb{N}$ is the set of all natural numbers and $\leq$, the relation of *less than or equal to* with usual meaning, is an order on $\mathbb{N}$. Clearly, the order $\leq$ consists of all the order pairs $(x, y) \in \mathbb{N}^2$ such that $x \leq y$ in $\mathbb{N}$.

(4) $\langle \mathbb{I}, | \rangle$, where $\mathbb{I}$ is the set of all prime numbers and $|$, the relation of *divisibility*, is an order on $\mathbb{I}$. Obviously, the order $|$ consists of only the ordered pairs $(x, x) \in \mathbb{I}^2$ for all $x \in \mathbb{I}$.

We assume every poset is finite and nonempty. Also, we use the notation **1** for the singleton poset $\mathbf{P} = \langle P, \leqslant \rangle$, where $P = \{x\}$ and $\leqslant = \{(x, x)\}$.

2.1.2. Comparable and incomparable elements.

Definition 2.1.2 Let $\mathbf{P} = \langle P, \leqslant \rangle$ be a poset. For $x, y \in P$, we say that x and y are *comparable* if either $x \leqslant y$ or $y \leqslant x$. Otherwise, we say that x and y are *incomparable* and we write $x \parallel y$.

Examples 2.1.2

(1) Every two elements in the poset $\langle \mathbb{N}, \leq \rangle$ (Example 2.1.1) are comparable.

(2) Every two distinct elements in the poset $\langle \mathbb{I}, | \rangle$ (Example 2.1.1) are incomparable.

2.1.3. Chain and antichain posets.

Definition 2.1.3 A poset $\mathbf{P} = \langle P, \leqslant_P \rangle$ is called a *chain* if x and y are comparable for every $x, y \in P$. On the other hand, $\mathbf{P}$ is called an *antichain* if $x \parallel y$ for every $x, y \in P$.
We use the notations $\mathbf{C}_n$ $(n \geq 1)$ for the n-element chain and $\mathbf{I}_n$ $(n \geq 1)$ for the n-element antichain.

Examples 2.1.3

(1) Trivially, the singleton poset **1** is both a chain and an antichain.

(2) The poset $\langle D(p), | \rangle$ (Example 2.1.1), where p is a prime number, is a 2-element chain.

(3) The poset $\langle \mathbb{N}, \leq \rangle$ (Example 2.1.1) is an infinite chain.

(4) The poset $\langle \mathbb{I}, | \rangle$ (Example 2.1.1) is an infinite antichain.

2.1.4. Covers of an element.

Definition 2.1.4 Let $\mathbf{P} = \langle P, \leqslant \rangle$ be a poset. For $x, y \in P$, we say that x *is covered by* y (or y *covers* x) if $x \leqslant y$ and $x \leqslant z \leqslant y$ implies either $x = z$ or $y = z$ for all $z \in P$.
We write $x \prec y$ (or $y \succ x$) if x is covered by y (or y covers x).

Examples 2.1.4

(1) In $\langle \mathbb{N}, \leq \rangle$ (Example 2.1.1), we have $n \prec n + 1$ for every $n \in \mathbb{N}$.
(2) In $\langle D(6), \leq \rangle$ (Example 2.1.1), we have $1 \prec 2 \prec 6$ and $1 \prec 3 \prec 6$.

2.1.5. Maximal and minimal elements.

Definition 2.1.5 Let $\mathbf{P} = \langle P, \leqslant \rangle$ be a poset. An element $x \in P$ is called a *maximal* element if for every $y \in P$, $x \leqslant y$ implies $x = y$. Analogously, an element $x \in P$ is called a *minimal* element if for every $y \in P$, $y \leqslant x$ implies $x = y$.
We use the notation $\max(P)$ for the set of all maximal elements, and $\min(P)$ for the set of all minimal elements of the poset $\mathbf{P} = \langle P, \leqslant \rangle$. If $x \in \max(P)$ is unique then we call x the *maximum* element of $\mathbf{P}$. Analogously, if $x \in \min(P)$ is unique then we call x the *maximum* element of $\mathbf{P}$.

Examples 2.1.5

(1) Trivially, every element of an antichain is both a maximal element and a minimal element.
(2) In the poset $\langle D(12), \leq \rangle$ (Example 2.1.1), the elements 1 and 12 are the minimum element and the maximum element, respectively.
(3) The infinite chain $\langle \mathbb{N}, \leq \rangle$ (Example 2.1.1) has the minimum element 1 and has no maximal element.

2.1.6. Hasse diagram.

One of the most useful and attractive features of the posets is that, in the finite cases, we can represent these by diagrams. Let $\mathbf{P}$ be a finite poset. We represent $\mathbf{P}$ by a configuration of circles (preferably, filled circles) representing the elements of **P** and interconnecting lines indicating the covering relation. Such a diagram of a poset is known as the *Hasse diagram* or *directed covering graph* or *digraph*. The construction of the Hasse diagram of a poset **P** goes as follows.

(1) To each point $x \in P$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle (preferably, filled circle) with center at $p(x)$.

(2) For each covering pair $(x, y) \in P$, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.

(3) Carry out (1) and (2) in such a way that if $x \leqslant y$, then $p(x)$ is lower than $p(y)$ and the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

Example 2.1.1 Hasse diagrams of the n-element chain $\mathbf{C}_n$, $n \geq 1$ and the n-element antichain $\mathbf{I}_n$, $n \geq 1$ are given in Figure 2.1.

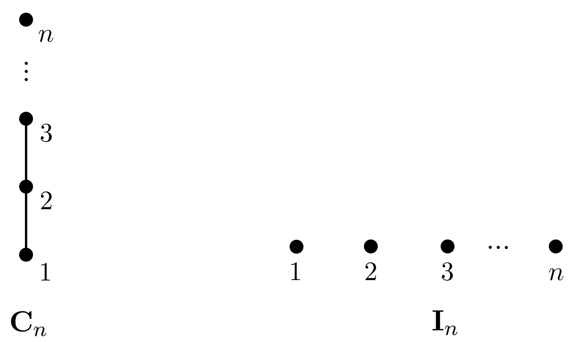


FIGURE 2.1. Hasse diagrams of the chain $\mathbf{C}_n$, $n \geq 1$ and the antichain $\mathbf{I}_n$, $n \geq 1$.

Example 2.1.2 Hasse diagrams of the poset $\langle P(\{x, y, z\}), \subseteq \rangle$ (Example 2.1.1) and the poset $\langle D(12), | \rangle$ (Example 2.1.1) are given in Figure 2.2.

2.1.7. Subposet.

Definition 2.1.6 Let $\mathbf{A} = \langle A, \leqslant \rangle$ be a poset and $\emptyset \neq B \subseteq A$. Then the poset $\mathbf{B} = \langle B, \leqslant \rangle$, the set B with the induced order in $\mathbf{A}$, is called a *subposet* or *subordered set* of $\mathbf{A}$.

A subposet $\mathbf{B} = \langle B, \leqslant \rangle$ of $\mathbf{A} = \langle A, \leqslant \rangle$ is called *proper* if $B \subset A$.

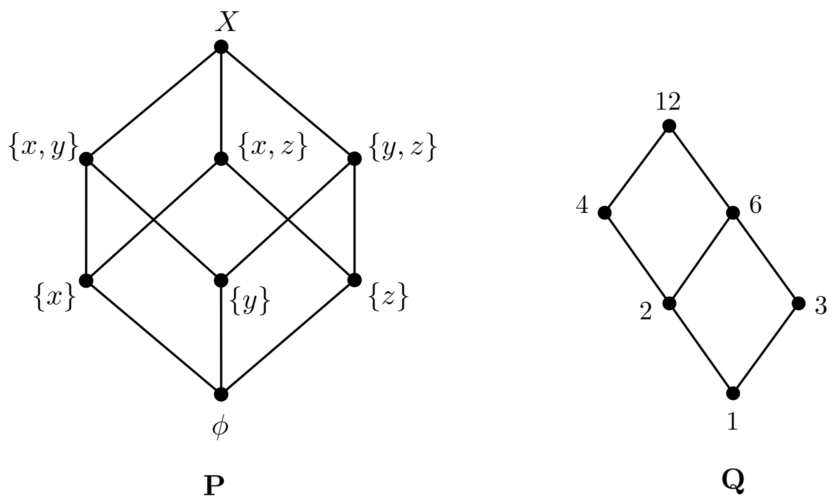


FIGURE 2.2. Hasse diagrams of $\mathbf{P} = \langle P(\{x, y, z\}), \subseteq\rangle$ and $\mathbf{Q} = \langle D(12), |\rangle$ (Example 2.1.1).

Examples 2.1.6

(1) The poset $\langle P(\{x, y\}), \subseteq\rangle$ (Example 2.1.1) is a subposet of the poset $\langle P(\{x, y, z\}), \subseteq\rangle$ (see the poset **P** in Figure 2.2).

(2) The poset $\langle D(6), |\rangle$ (Example 2.1.1) is a subposet of the poset $\langle D(12), |\rangle$ (see the poset **Q** in Figure 2.2).

Remark 2.1.1 Every subposet of a chain (analogously, antichain) is also a chain (analogously, antichain).

2.1.8. Connected and disconnected posets.

Definition 2.1.7 A poset $\mathbf{A} = \langle A, \leqslant\rangle$ is called *connected* if there do not exist the subposets $\mathbf{B} = \langle B, \leqslant\rangle$ and $\mathbf{C} = \langle C, \leqslant\rangle$ of **A** such that $A = B\dot{\cup}C$ and $x \parallel y$ for every $x \in B$ and $y \in C$. Otherwise, **A** is called *disconnected* and then the subposets **B** and **C** are called the *components* of the disconnected poset **A**.

Examples 2.1.7

(1) The singleton poset **1** is connected trivially.

(2) For every $n \geq 1$, the chain $\mathbf{C}_n$ is connected.

(3) For every $n \geq 2$, the antichain $\mathbf{I}_n$ is disconnected. Here for every $m < n$, the poset $\mathbf{I}_m$ is a component of $\mathbf{I}_n$.

2.1.9. Order isomorphism.

Definition 2.1.8 Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be two posets. Then a bijective map $\phi : A \to B$ is called an *order isomorphism* if for all $x, y \in A$,

$$x \leqslant_A y \text{ if and only if } \phi(x) \leqslant_B \phi(y).$$

If an order isomorphism $\phi : A \to B$ exists then we say that $\mathbf{A}$ and $\mathbf{B}$ are *order isomorphic* and we write $\mathbf{A} \cong \mathbf{B}$.

Examples 2.1.8

(1) Trivially, $\mathbf{C}_1 \cong \mathbf{I}_1 \cong \mathbf{1}$.

(2) Consider the map $\phi : \langle D(12), | \rangle \to \langle D(20), | \rangle$ defined as $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 5$, $\phi(4) = 4$, $\phi(6) = 10$, and $\phi(12) = 20$. Then ϕ is an order isomorphism and hence $\langle D(12), | \rangle \cong \langle D(20), | \rangle$.

2.1.10. Dual poset.

Definition 2.1.9 Let $\mathbf{P} = \langle P, \leqslant \rangle$ be a poset. The *dual order* of $\leqslant$ on P, denoted by $\leqslant^\partial$, is defined as follows:

$$x \leqslant^\partial y \text{ if and only if } y \leqslant x \text{ for all } x, y \in P.$$

The *dual poset* of $\mathbf{P}$, denoted by $\mathbf{P}^\partial$, is defined as the poset with the dual order $\leqslant^\partial$ on P, that is, $\mathbf{P}^\partial = \langle P, \leqslant^\partial \rangle$.

Examples 2.1.9

(1) For every $n \geq 1$, trivially, $\mathbf{C}_n^\partial \cong \mathbf{C}_n$ and $\mathbf{I}_n^\partial \cong \mathbf{I}_n$.

(2) We have $\langle D(12), |^\partial \rangle \cong \langle D(12), | \rangle$, where for every $x, y \in D(12)$, $x|^\partial y$ if and only if x is divisible by y.

2.2. Common posets

2.2.1. Complete bipartite poset.

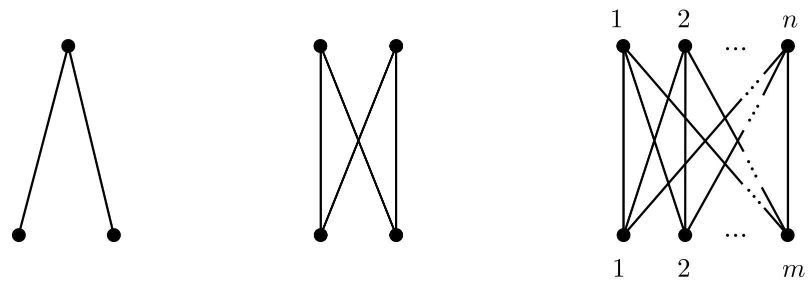
Definition 2.2.1 An $(m+n)$-element poset $\mathbf{P}=\langle P;\leqslant_P\rangle$, where $m \geq n \geq 1$, is called a *complete bipartite* poset if and only if the following conditions hold:

(1) $|\min(P)| = m$ and $|\max(P)| = n$,

(2) $x \prec y$, for all $x \in \min(P)$, $y \in \max(P)$.

We use the notation $\mathbf{B}_{m,n}$ $(m \geq n \geq 1)$ for the complete bipartite poset with m minimal elements and n maximal elements.

Example 2.2.1 The complete bipartite posets $\mathbf{B}_{2,1}$, $\mathbf{B}_{2,2}$, and $\mathbf{B}_{m,n}$, where $m \geq n \geq 1$, are shown in Figure 2.3.



$\mathbf{B}_{2,1}$ $\mathbf{B}_{2,2}$ $\mathbf{B}_{m,n}$

FIGURE 2.3. Hasse diagrams of the complete bipartite posets $\mathbf{B}_{2,1}$, $\mathbf{B}_{2,2}$, and $\mathbf{B}_{m,n}$ $(m \geq n \geq 1)$.

Remarks 2.2.1

(1) $\mathbf{B}_{m,n} \cong \mathbf{B}_{n,m}$ for every $m = n \geq 1$.

(2) $\mathbf{B}^{\partial}_{m,n} \cong \mathbf{B}_{n,m}$ for every $m \geq n \geq 1$.

2.2.2. Zigzag poset.

Definition 2.2.2 An n-element poset $\mathbf{P}=\langle P;\leqslant_P\rangle$, where $n \geq 4$, is called a *zigzag* or *fence* if and only if the following conditions hold:

(1) $|\min(P)| = \lceil \frac{n}{2} \rceil$ and $|\max(P)| = \lfloor \frac{n}{2} \rfloor$,

(2) $x_i \prec y_i$, for every $x_i \in \min(P)$, $y_i \in \max(P)$ where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

(3) $x_i \prec y_{i-1}$, for every $x_i \in \min(P)$, $y_i \in \max(P)$ where $2 \leq i \leq \lceil \frac{n}{2} \rceil$.

We use the notation $\mathbf{Z}_n$ ($n \geq 4$) for the n-element zigzag or n-element fence.

Example 2.2.2 The zigzag posets $\mathbf{Z}_4$, $\mathbf{Z}_5$, and $\mathbf{Z}_n$, where $n \geq 4$, are shown in Figure 2.4.

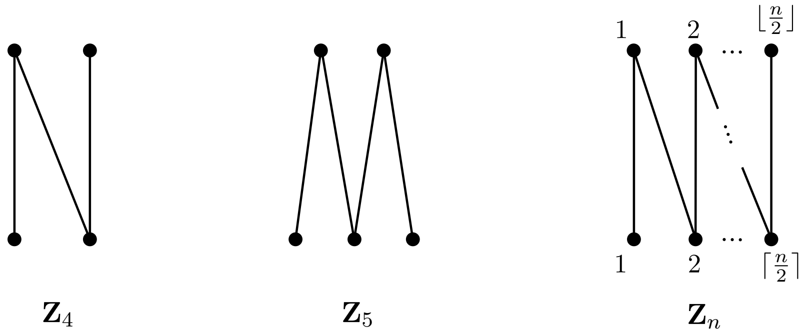


FIGURE 2.4. Hasse diagrams of the zigzag posets $\mathbf{Z}_4$, $\mathbf{Z}_5$, and $\mathbf{Z}_n$ ($n \geq 4$).

Remark 2.2.1 For every even $n \geq 4$, we have $\mathbf{Z}_n^\partial \cong \mathbf{Z}_n$.

2.2.3. Diamond poset.

Definition 2.2.3 A *diamond* poset is an n-element poset $\mathbf{P}{=}\langle P; \leqslant_P \rangle$, where $n \geq 4$, that consists of $n-2$ distinct 3-element chains as subposets and has exactly one minimal element and one maximal element.

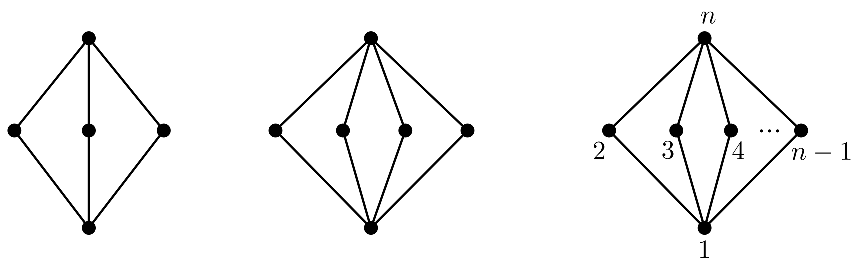
We use the notation $\mathbf{D}_n$ ($n \geq 4$) for the n-element diamond poset.

Example 2.2.3 The diamond posets $\mathbf{D}_5$, $\mathbf{D}_6$, and $\mathbf{D}_n$, where $n \geq 4$, are shown in Figure 2.5.

Remarks 2.2.2

(1) $\mathbf{D}_4 \cong \langle D(6), | \rangle$, as in Example 2.1.1.

(2) For every $n \geq 4$, clearly, $\mathbf{D}_n^\partial \cong \mathbf{D}_n$.



$\mathbf{D}_5$ $\mathbf{D}_6$ $\mathbf{D}_n$

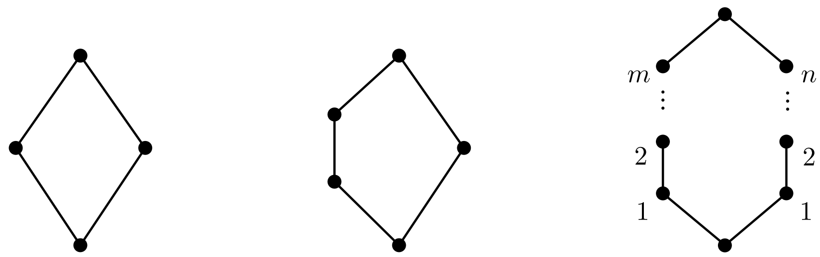
FIGURE 2.5. Hasse diagrams of the diamond posets $\mathbf{D}_5$, $\mathbf{D}_6$, and $\mathbf{D}_n$ ($n \geq 4$).

2.2.4. Polygonal poset.

Definition 2.2.4 A *polygonal* poset is an $(m+n+2)$-element connected poset $\mathbf{P}$=$\langle P; \leqslant_P \rangle$, where $m \geq n \geq 1$, that has exactly two distinct chains as the subposets consisting of $m+2$ and $n+2$ elements, respectively, with only the minimum and the maximum elements in common.

We use the notation $\mathbf{P}_{m,n}$ ($m \geq n \geq 1$) for the $(m+n+2)$-element polygonal poset.

Example 2.2.4 The polygonal posets $\mathbf{P}_{1,1}$, $\mathbf{P}_{2,1}$, and, in general, $\mathbf{P}_{m,n}$, where $m \geq n \geq 1$, are shown in Figure 2.6.



$\mathbf{P}_{1,1}$ $\mathbf{P}_{2,1}$ $\mathbf{P}_{m,n}$

FIGURE 2.6. Hasse diagrams of the polygonal posets $\mathbf{P}_{1,1}$, $\mathbf{P}_{2,1}$, and $\mathbf{P}_{m,n}$ ($m \geq n \geq 1$).

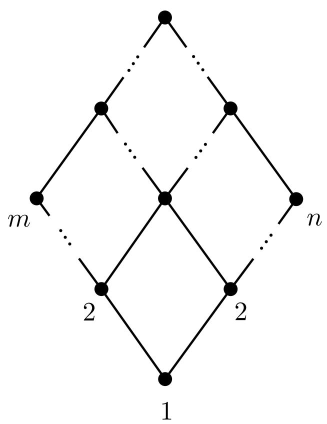
Remarks 2.2.3

(1) $\mathbf{P}_{1,1} \cong \mathbf{D}_4$.

(2) $\mathbf{P}^{\partial}_{m,n} \cong \mathbf{P}_{m,n}$ for every $m \geq n \geq 1$.

2.2.5. Ladder poset.

Definition 2.2.5 An m-by-n *ladder* poset, denoted by $\mathbf{L}_{m,n}$, where $m \geq n \geq 2$, is an $(m \times n)$-element poset that is isomorphic to the poset shown in Figure 2.7 by using the Hasse diagram.
In particular, we use the notation $\mathbf{L}_m$ for $\mathbf{L}_{m,2}$, where $m \geq 2$.



$\mathbf{L}_{m,n}$

FIGURE 2.7. Hasse diagram of $\mathbf{L}_{m,n}$ $(m \geq n \geq 2)$.

Example 2.2.5 In particular, the ladders $\mathbf{L}_{3,2}$ and $\mathbf{L}_{m,2}$, where $m \geq 2$, are shown in Figure 2.8.

Remarks 2.2.4

(1) $\mathbf{L}_2 \cong \mathbf{P}_{1,1} \cong \mathbf{D}_4$.

(2) $\mathbf{L}_3 \cong \langle D(12), | \rangle$ (Example 2.1.1).

(3) $\mathbf{L}_{(d_1+1),(d_2+1)} \cong \langle D(n_1^{d_1} n_2^{d_2}), | \rangle$, where n_1 and n_2 are primes, and d_1 and d_2 are positive integers.

(4) $\mathbf{L}^{\partial}_{m,n} \cong \mathbf{L}_{m,n}$ for every $m \geq n \geq 2$.

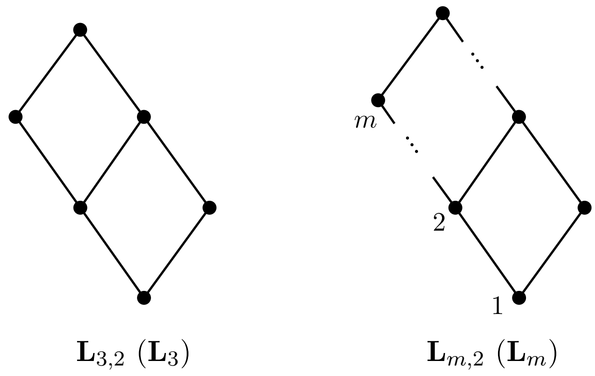


FIGURE 2.8. Hasse diagrams of the ladders $\mathbf{L}_{3,2}$ and $\mathbf{L}_{m,2}$ ($m \geq 2$).

2.2.6. Height-balanced tree poset.

Definition 2.2.6 A t-ary n-element *height-balanced rooted tree*, where $n > t \geq 1$, denoted by $\mathbf{T}_{t,n}$, is a poset that is isomorphic to the poset shown in Figure 2.9.

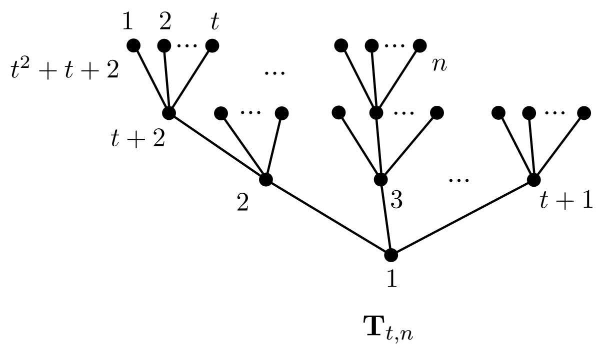


FIGURE 2.9. Hasse diagram of $\mathbf{T}_{t,n}$.

Example 2.2.6 In particular, the height-balanced rooted trees $\mathbf{T}_{2,4}$ and $\mathbf{T}_{3,9}$ are shown in Figure 2.10.

Remarks 2.2.5

(1) $\mathbf{T}_{1,2} \cong \mathbf{B}_{1,1} \cong \mathbf{C}_2$.

(2) $\mathbf{T}_{1,n} \cong \mathbf{C}_n$ for every $n \geq 2$.

(3) $\mathbf{T}_{t,t+1} \cong \mathbf{B}_{1,t}$ for every $t \geq 1$.

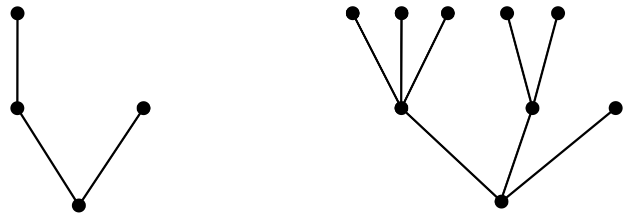


FIGURE 2.10. Hasse diagrams of the trees $\mathbf{T}_{2,4}$ and $\mathbf{T}_{3,9}$.

2.3. Basic operations

2.3.1. Direct sum of posets.

Definition 2.3.1 Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be posets on the disjoint sets A and B. Then the *direct sum* (*disjoint sum* or *free sum* or *parallel composition*) of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is defined as the poset $\langle A \cup B, \leqslant_+ \rangle$ such that for every $x, y \in A \cup B$,

$$x \leqslant_+ y \text{ if and only if } x \leqslant_A y \text{ or } x \leqslant_B y.$$

Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *direct terms* (*direct components*) of the poset $\mathbf{A} + \mathbf{B}$.

Note that, a poset having two or more components is called disconnected and, otherwise, it is called connected. These give the alternative definitions of the connected and disconnected posets.

Example 2.3.1 The direct sums $\mathbf{B}_{2,1} + \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$ are shown in Figure 2.11.

Remarks 2.3.1

(1) For every $n \geq 2$, we have $\mathbf{I}_n \cong \underbrace{\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}}_{n \text{ times}}$. Thus, antichains are the direct sums of the singleton posets.

(2) Let $\mathbf{A}$ be a disconnected poset having n components $\mathbf{A}_i$, $1 \leq i \leq n$. Then $\mathbf{A} \cong \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n$.

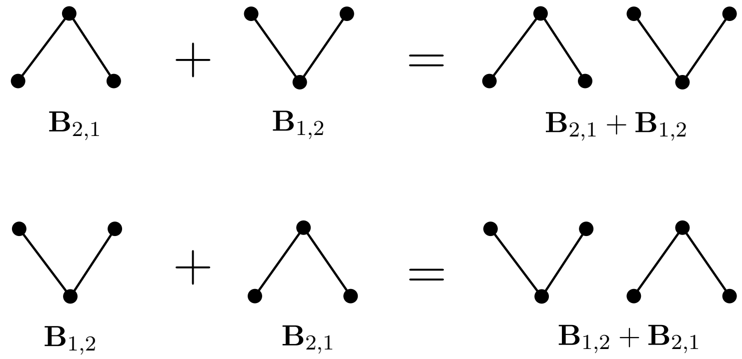


FIGURE 2.11. Hasse diagrams showing $\mathbf{B}_{2,1} + \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$.

(3) We observe that $\mathbf{B}_{2,1} + \mathbf{B}_{1,2} \cong \mathbf{B}_{1,2} + \mathbf{B}_{2,1}$ (Example 2.3.1). In general, for any posets $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A}+\mathbf{B} \cong \mathbf{B}+\mathbf{A}$. Thus, the direct sum of posets is commutative.

(4) Posets obtained as the direct sum of two or more posets are disconnected.

Note 2.3.1 For any posets $\mathbf{A}_i, 1 \leq i \leq n$, we write shortly $\sum_{i=1}^{n} \mathbf{A}_i$ for the direct sum $\mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n$. In particular, we write shortly $n\mathbf{A}$ for the direct sum $\mathbf{A} + \mathbf{A} + \cdots + \mathbf{A}$ of the n posets $\mathbf{A}$.

2.3.2. Ordinal sum of posets.

Definition 2.3.2 Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be posets on the disjoint sets A and B. Then the *ordinal sum* (*linear sum* or *series composition*) of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \oplus \mathbf{B}$, is defined as the poset $\langle A \cup B, \leqslant_\oplus \rangle$ such that for every $x, y \in A \cup B$,

$$x \leqslant_\oplus y \text{ if and only if } x \leqslant_A y \text{ or } x \leqslant_B y, \text{ or } x \in A \text{ and } y \in B.$$

Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *ordinal terms* of $\mathbf{A} \oplus \mathbf{B}$.

Example 2.3.2 The ordinal sums $\mathbf{B}_{2,1} \oplus \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ are shown in Figure 2.12.

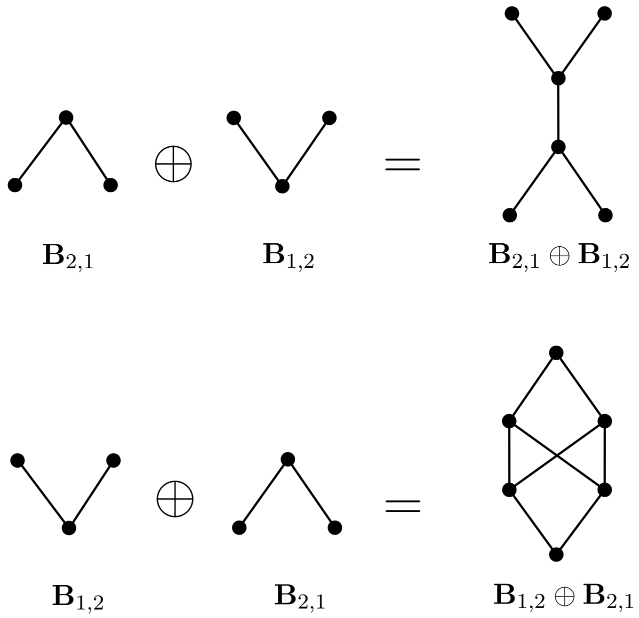


FIGURE 2.12. Hasse diagrams showing $\mathbf{B}_{2,1} \oplus \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$.

Remarks 2.3.2

(1) For every $n \geq 2$, we have $\mathbf{C}_n \cong \underbrace{\mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{n \text{ times}}$. Thus, chains are the ordinal sums of the singleton posets.

(2) For every $m \geq n \geq 1$, we have $\mathbf{B}_{m,n} \cong \mathbf{I}_m \oplus \mathbf{I}_n$. Thus, complete bipartite posets are the ordinal sums of the antichains.

(3) We observe that $\mathbf{B}_{2,1} \oplus \mathbf{B}_{1,2} \ncong \mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ (Example 2.3.2). In general, for any posets $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \oplus \mathbf{B} \ncong \mathbf{B} \oplus \mathbf{A}$. This shows that the ordinal sum of posets is not commutative.

(4) The posets obtained as the ordinal sum of two or more posets are connected.

Note 2.3.2 For any posets $\mathbf{A}_i, 1 \leq i \leq n$, we write shortly $\bigoplus_{i=1}^{n} \mathbf{A}_i$ for the ordinal sum $\mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n$. In particular, we write shortly $\oplus^n \mathbf{A}$ for the ordinal sum $\mathbf{A} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}$ of the n posets $\mathbf{A}$.

2.3.3. Direct product of posets.

Definition 2.3.3 Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be posets on the disjoint sets A and B. Then the *direct product* of posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \times \mathbf{B}$, is defined as the poset $\langle A \times B, \leqslant \rangle$ such that for every $(x, y), (x', y') \in A \times B$,

$$(x, y) \leqslant (x', y') \text{ if and only if } x \leqslant_A x' \text{ and } y \leqslant_B y'.$$

Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *direct factors* of $\mathbf{A} \times \mathbf{B}$.

Example 2.3.3 The direct products $\mathbf{B}_{\mathbf{1,2}} \times \mathbf{B}_{\mathbf{2,1}}$ and $\mathbf{B}_{\mathbf{2,1}} \times \mathbf{B}_{\mathbf{1,2}}$ are shown in Figure 2.13.

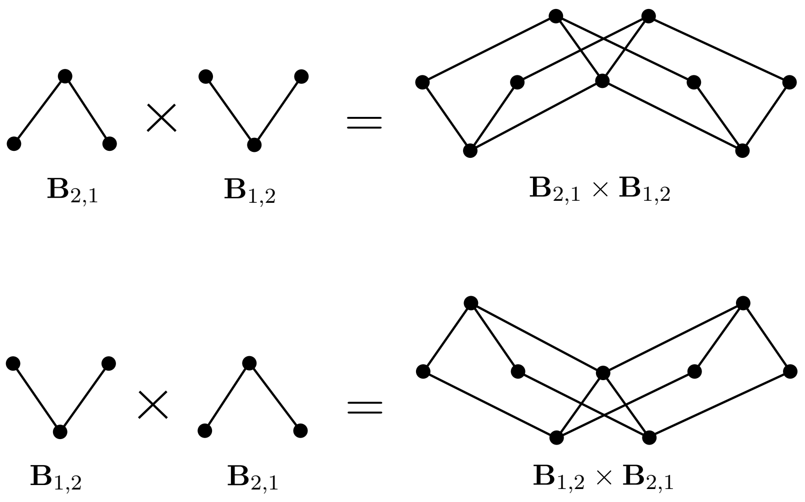


FIGURE 2.13. Hasse diagrams showing $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \times \mathbf{B}_{2,1}$.

Remarks 2.3.3

(1) Let $\mathbf{A}$ be any poset. Then $\mathbf{I}_n \times \mathbf{A} \cong n\mathbf{A}$.

(2) For every $m \geq n \geq 2$, we have $\mathbf{C}_m \times \mathbf{C}_n \cong \mathbf{L}_{m,n}$.

(3) We observe that $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2} \cong \mathbf{B}_{1,2} \times \mathbf{B}_{2,1}$ (Example 2.3.3). In general, for any posets $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \times \mathbf{B} \cong \mathbf{B} \times \mathbf{A}$ implying that the direct product of posets is commutative.

Note 2.3.3 For any posets $\mathbf{A}_i, 1 \leq i \leq n$, we write shortly $\prod_{i=1}^{n} \mathbf{A}_i$ for the direct product $\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n$. In particular, we write shortly $\mathbf{A}^n$ for the direct product $\mathbf{A} \times \mathbf{A} \times \cdots \times \mathbf{A}$ of the n posets $\mathbf{A}$.

2.3.4. Ordinal product of posets.

Definition 2.3.4 Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be posets on the disjoint sets A and B. The *ordinal product* of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as the poset $\langle A \times B, \leqslant_\otimes \rangle$ such that for every $(x, y), (x', y') \in A \times B$,

$$(x, y) \leqslant_\otimes (x', y') \text{ if and only if } x \leqslant_A x' \text{ or } x = x' \text{ implies } y \leqslant_B y'.$$

Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *ordinal factors* of $\mathbf{A} \otimes \mathbf{B}$.

Example 2.3.4 The ordinal products $\mathbf{B_{1,2}} \otimes \mathbf{B_{2,1}}$ and $\mathbf{B_{2,1}} \otimes \mathbf{B_{1,2}}$ are shown in Figure 2.14.

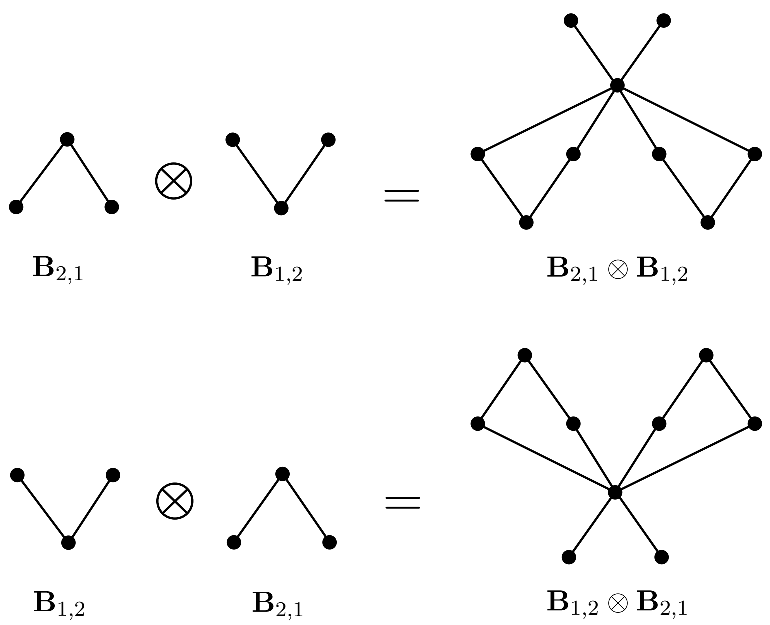


FIGURE 2.14. Hasse diagrams showing $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$.

Remarks 2.3.4

(1) For any poset $\mathbf{A}$, we have $\mathbf{I}_n \otimes \mathbf{A} \cong n\mathbf{A}$.

(2) For any poset $\mathbf{B}$, we have $\mathbf{C}_n \otimes \mathbf{B} \cong \oplus^n \mathbf{B}$. In Section 3.6, we prove this result by using the poset matrix.

(3) We see that $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2} \ncong \mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$ (Example 2.3.4). In general, $\mathbf{A} \otimes \mathbf{B} \ncong \mathbf{B} \otimes \mathbf{A}$ for some posets $\mathbf{A}$ and $\mathbf{B}$. This shows that the ordinal product of posets is not commutative.

2.3.5. Composition of posets.

Definition 2.3.5 Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ with $A = \{x_1,\ x_2,\ ...,\ x_m\}$ and $\mathbf{B}_r = \langle B_r, \leqslant_{B_r} \rangle$, $1 \leq r \leq m$ with $B_r = \{y_{t+i} : 1 \leq i \leq n_r\}$ where $t = \sum_{k=1}^{r-1} n_k$, be posets on the disjoint sets A and B_r, $1 \leq r \leq m$, respectively. Then the *composition* of the posets $\mathbf{A}$ and $\mathbf{B}_r$, $1 \leq r \leq m$, denoted by $\mathbf{A}\ [\mathbf{B}_1,\ \mathbf{B}_2,\ ...,\ \mathbf{B}_m]$, is defined as the poset $\langle \bigcup_{k=1}^{m} B_k, \leqslant_c \rangle$ such that for every $y_i, y_j \in \bigcup_{r=1}^{m} B_r$, we have $y_i \leqslant_c y_j$ if and only if one of the following holds:

(1) $y_{t+i'}, y_{l+j'} \in B_r$ for some r (when $t = l = \sum_{k=1}^{r-1} n_k$, $i' = i - t$ and $j' = j - l$) and $y_{t+i'} \leqslant_{B_r} y_{l+j'}$,

(2) $y_{t+i'} \in B_r$ and $y_{l+j'} \in B_s$ for some $r < s$ (when $\sum_{k=1}^{r-1} n_k = t < l = \sum_{k=1}^{s-1} n_k$, $i' = i - t$ and $j' = j - l$) and $x_r \leqslant_A x_s$.

Here, $\mathbf{A}$ is called the *outer poset* or *quotient poset*, and $\mathbf{B}_r, 1 \leq r \leq m$ are called *inner posets* and their ground sets are called *autonomous sets*.

Example 2.3.5 The composition $\mathbf{B}_{2,1}[\mathbf{C}_2,\ \mathbf{Z}_4,\ \mathbf{B}_{1,2}]$ of the posets $\mathbf{B}_{2,1}$, $\mathbf{C}_2$, $\mathbf{Z}_4$, and $\mathbf{B}_{1,2}$ is shown in Figure 2.15. Here the poset $\mathbf{B}_{2,1}$ is the outer poset and the posets $\mathbf{C}_2$, $\mathbf{Z}_4$, and $\mathbf{B}_{1,2}$ are the inner posets.

Remarks 2.3.5

(1) For any n-element poset $\mathbf{A}$, clearly, $\mathbf{A}[\underbrace{\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}}_{n \text{ times}}] \cong \mathbf{A}$.

(2) For a collection of posets $\mathbf{B}_i$, $1 \leq i \leq n$, we have $\mathbf{I}_n[\mathbf{B}_1,\ \mathbf{B}_2,\ \ldots,\ \mathbf{B}_n] \cong \sum_{i=1}^{n} \mathbf{B}_i$ and $\mathbf{C}_n[\mathbf{B}_1,\ \mathbf{B}_2,\ \ldots,\ \mathbf{B}_n] \cong \bigoplus_{i=1}^{n} \mathbf{B}_i$.

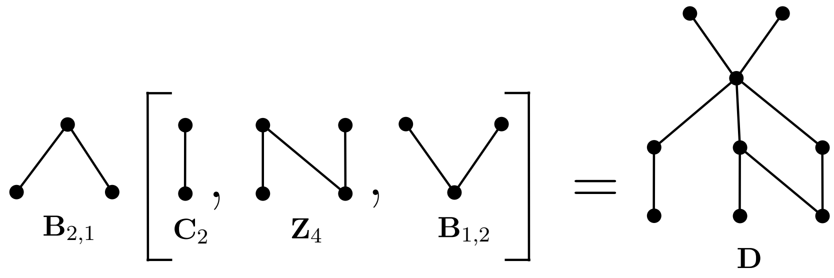


FIGURE 2.15. Hasse diagrams showing $\mathbf{D} = \mathbf{B}_{2,1}[\mathbf{C}_2, \mathbf{Z}_4, \mathbf{B}_{1,2}]$.

2.4. Decomposable posets

The class of all decomposable posets is the class of mostly computational tractable posets. For immediate simplicity, methods for solving many optimization problems on structure theory begin with some decomposition techniques. These techniques are used to reduce a bigger structure into smaller ones of the same kind, like posets into autonomous sets [**27, 31**], graphs into clumps [**5**], comparability graphs into stable sets [**50**], schedules into job-modules [**36**], networks into simplifiable subnetworks [**51**] and so on. As a result, several subclasses of the decomposable posets are considered in the literature by numerous authors. Some more details on the classes of computationally tractable posets was studied by Möhring [**40**].

2.4.1. Decomposable poset.

Definition 2.4.1 A poset $\mathbf{D}$ is called *decomposable* if it can be obtained as the composition of two or more inner posets where at least one inner poset is nonsingleton. Thus, a poset $\mathbf{D}$ is decomposable if and only if there exist the poset $\mathbf{A}$ and the posets $\mathbf{B}_1$, $\mathbf{B}_2$, ..., $\mathbf{B}_n$, $n \geq 2$, where at least one $\mathbf{B}_i$ is nonsingleton, such that $\mathbf{D} \cong \mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n]$.
A poset is called *prime* (*indecomposable*) if and only if it is not decomposable.

Examples 2.4.1

(1) Trivially, all the antichain posets $\mathbf{I}_n$, $n \geq 3$ are decomposable, because $\mathbf{I}_n \cong \mathbf{I}_2[\mathbf{1}, \mathbf{I}_{n-1}]$, where the poset $\mathbf{I}_{n-1}$ is nonsingleton.

(2) For every $n \geq 4$, the diamond poset $\mathbf{D}_n$ is decomposable, because $\mathbf{D}_n \cong \mathbf{C}_2[\mathbf{1}, \mathbf{B}_{(n-2),1}] \cong \mathbf{C}_2[\mathbf{B}_{1,(n-2)},\ \mathbf{1}] \cong \mathbf{C}_3[\mathbf{1},\ \mathbf{I}_{n-2},\ \mathbf{1}]$.

(3) For every $m, n \geq 1$, the complete bipartite poset $\mathbf{B}_{m,n}$ is decomposable, because $\mathbf{B}_{m,n} \cong \mathbf{C}_2[\mathbf{I}_m, \mathbf{I}_n]$.

(4) Let $\mathbf{A}$ be a disconnected poset consisting of the connected components $\mathbf{A}_i$, $1 \leq i \leq n$, where $n \geq 2$ such that at least one $\mathbf{A}_i$ is nonsingleton. Then $\mathbf{A}$ is decomposable, because $\mathbf{A} \cong \mathbf{I}_n[\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$.

(5) The zigzag poset $\mathbf{Z}_4$ is the simplest example of a prime poset.

Remarks 2.4.1 We see that if $\mathbf{D}$ is a decomposable poset then $|D| \geq 3$, and therefore, the posets $\mathbf{1}$, $\mathbf{I}_2$, and $\mathbf{C}_2$ are not decomposable. Here, we assume that these posets are trivially decomposable, because it does not affect the purpose of the property of decomposition of posets. Further, we see that this assumption helps generalizing some important results related to the recognition of posets.

2.4.2. *P*-graph.

Definition 2.4.2 A poset $\mathbf{G}$ is called a *P-graph* if either it is an antichain poset or it can be expressed as the ordinal sum of the antichain posets. In other words, $\mathbf{G}$ is a *P-graph* if there exist the antichain posets $\mathbf{A}_i, 1 \leq i \leq n$ such that $\mathbf{G} \cong \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n \cong \bigoplus_{i=1}^{n} \mathbf{A}_i$.

Examples 2.4.2

(1) Trivially, all the antichain posets $\mathbf{I}_n$, $n \geq 1$ are P-graphs.

(2) All the chain posets $\mathbf{C}_n$, $n \geq 2$ are P-graphs, because $\mathbf{C}_n \cong \oplus^n \mathbf{1} \cong \underbrace{\mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{n \text{ times}}$.

(3) Obviously, $\mathbf{B}_{m,n} \cong \mathbf{I}_m \oplus \mathbf{I}_n$ for every $m, n \geq 1$. Thus, all complete bipartite posets are P-graphs.

(4) We have $\mathbf{D}_n \cong \mathbf{1} \oplus \mathbf{I}_{n-2} \oplus \mathbf{1}$ for every $n \geq 4$. This shows that all diamond posets are P-graphs.

(5) All the posets except $\mathbf{C}_2 + \mathbf{1}$ having the number of elements less than or equal to 3 (shown in Figure 2.16 by using the Hasse diagrams) are P-graphs.

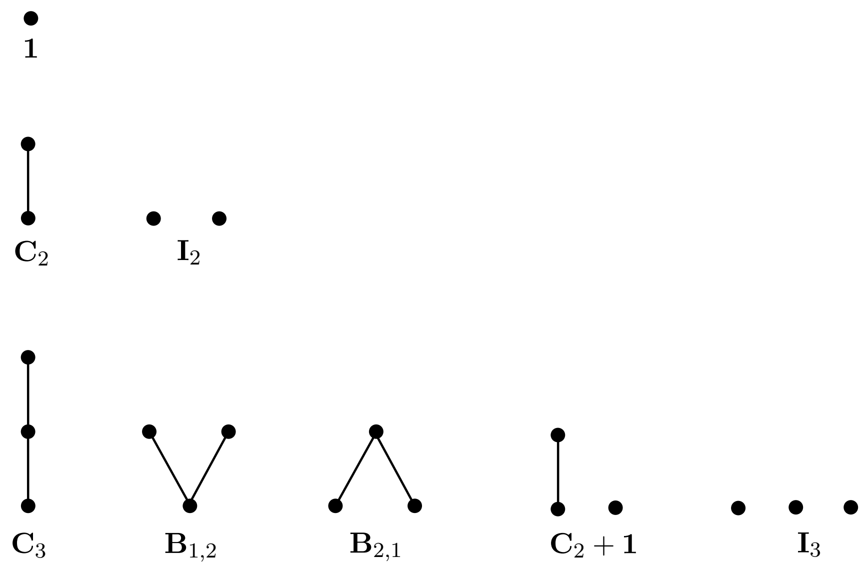


FIGURE 2.16. Hasse diagrams of the posets with elements less than or equal to 3.

Remarks 2.4.2

(1) Obviously, all the P-graphs except the antichain posets $\mathbf{I}_n$, $n \geq 2$ are connected posets.

(2) Let $\mathbf{G}$ be any P-graph. If $|G| < 3$ then, by our assumption, $\mathbf{G}$ is trivially decomposable. Also, if $\mathbf{G} \cong \mathbf{I}_n$ for some $n \geq 3$ then $\mathbf{G}$ is clearly decomposable. Otherwise, $\mathbf{G}$ is connected and for some $2 \leq n \leq |S|$, we have $\mathbf{G} \cong \mathbf{C}_n[\mathbf{I}_{m_1}, \mathbf{I}_{m_2}, \ldots, \mathbf{I}_{m_n}]$ for some m_i, $1 \leq i \leq n$, where at least one $\mathbf{I}_{m_i}$ is a nonsingleton poset. Thus, every P-graph is decomposable.

2.4.3. *P*-series.

Definition 2.4.3 A poset $\mathbf{S}$ is called a *P-series* if either $\mathbf{S}$ is a P-graph or it can be expressed as the direct sum of P-graphs. Thus, $\mathbf{S}$ is a *P-series* if there exist the P-graphs $\mathbf{G}_i$, $1 \leq i \leq n$ such that $\mathbf{S} \cong \mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_n \cong \sum_{i=1}^{n} \mathbf{G}_i$.

Examples 2.4.3

(1) Every P-graph is trivially a P-series.

(2) The poset $\mathbf{C}_m + \mathbf{C}_n$ is a P-series which is not a P-graph if either $m \geq 2$ or $n \geq 2$.

(3) All the posets except $\mathbf{1} \oplus (\mathbf{C}_2 + \mathbf{1})$, $(\mathbf{C}_2 + \mathbf{1}) \oplus \mathbf{1}$, and $\mathbf{Z}_4$ with the number of elements less than or equal to 4 (shown in Figure 2.16 and Figure 2.17 by using the Hasse diagrams) are P-series.

Remarks 2.4.3

(1) All the P-series except the connected P-graphs are disconnected.

(2) All the connected P-series are P-graphs and therefore these are decomposable by the remarks on the P-graphs.

(3) Let $\mathbf{S}$ be a disconnected P-series. If $|S| < 3$ then, by our assumption, $\mathbf{S}$ is decomposable. Otherwise, for some $2 \leq n \leq |S|$, we have $\mathbf{S} \cong \mathbf{I}_n[\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n]$, where $\mathbf{G}_i$, $1 \leq i \leq n$ are P-graphs such that at least one $\mathbf{G}_i$ is nonsingleton. Thus, every disconnected P-series is decomposable.

2.4.4. Series-parallel poset.

Definition 2.4.4 A poset $\mathbf{P}$ is called *series-parallel* if it can be obtained from the singleton poset $\mathbf{1}$ by using only the direct sum and the ordinal sum. In other words, $\mathbf{P}$ is series-parallel if $\mathbf{P} \cong \mathbf{P}_1 * \mathbf{P}_2 * \cdots * \mathbf{P}_n$, where for every $1 \leq i \leq n$, the poset $\mathbf{P}_i$ is a P-series, and the operation $*$ is either the direct sum or the ordinal sum.

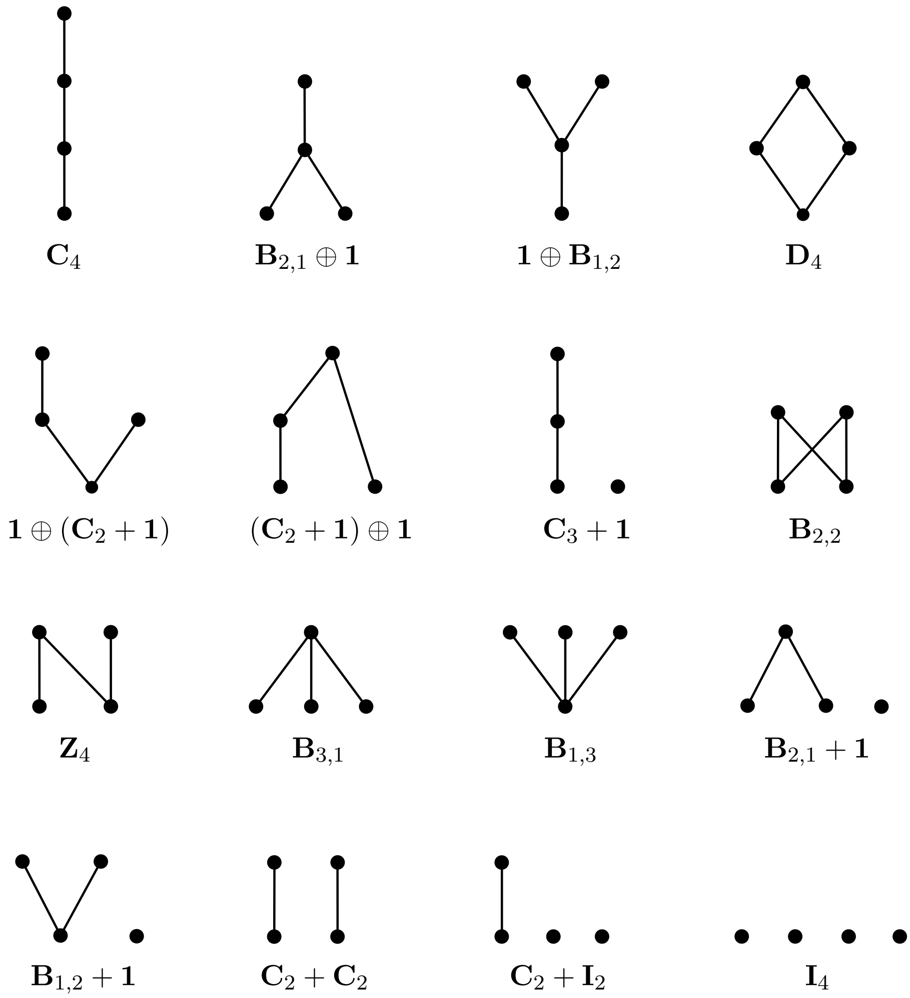


FIGURE 2.17. Hasse diagrams of all the 4-element posets.

Examples 2.4.4

(1) Every P-series as well as every P-graph is trivially series-parallel.

(2) We have $\mathbf{P}_{m,n} \cong \mathbf{1} \oplus (\mathbf{C}_m + \mathbf{C}_n) \oplus \mathbf{1}$, where $\mathbf{C}_r \cong \oplus^r \mathbf{1}$. This shows that $\mathbf{P}_{m,n}$, $m \geq n \geq 2$ can be obtained from the singleton poset using only the direct sum and ordinal sum. Thus, all the polygonal posets are series-parallel.

(3) We have $\mathbf{T}_{t,n} \cong r_0 \oplus (\mathbf{T}_{t,n_0} + \mathbf{T}_{t,n_0} + \cdots + \mathbf{T}_{t,n_0}) \cong r_0 \oplus (n\mathbf{T}_{t,n_0})$, where r_0 is the root (the singleton poset) of $\mathbf{T}_{t,n}$ and $\mathbf{T}_{t,n_0}$ is either a leaf

(the singleton poset) or a subtree with number of elements $n_0 = \frac{n-1}{t}$, particularly, if the tree $\mathbf{T}_{t,n}$ is full. This continues recursively expressing $\mathbf{T}_{t,n}$, $n \geq t+1 \geq 2$ as the sum of the singleton posets using only the direct sum and ordinal sum. This shows that height-balanced rooted trees are series-parallel.

Remarks 2.4.4

(1) The zigzag posets $\mathbf{Z}_n$, $n \geq 4$ are not series-parallel.

(2) Every direct term and ordinal term of a series-parallel poset is also series-parallel.

(3) In Section 3.7, we show by using the poset matrix that every series-parallel poset is decomposable. On the other hand, we see that the decomposable poset $\mathbf{Z}_4 \oplus \mathbf{1} \cong \mathbf{C}_2[\mathbf{Z}_4, \mathbf{1}]$ is not series-parallel, because $\mathbf{Z}_4$ is not a P-graph. Thus, a decomposable poset may not be series-parallel.

2.4.5. Factorable poset.

Definition 2.4.5 A poset $\mathbf{F}$ is said to be *factorable* if and only if it can be obtained as the direct product of two or more nonsingleton posets. In other words, $\mathbf{F}$ is *factorable* if and only if their exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{F} \cong \mathbf{A} \times \mathbf{B}$.

Examples 2.4.5

(1) Let $\mathbf{A}$ be a nonsingleton poset. For every $n \geq 2$, we have $n\mathbf{A} \cong \mathbf{I}_n \times \mathbf{A}$. This shows that every disconnected poset with pairwise isomorphic nonsingleton direct terms is trivially factorable.

(2) We have $\mathbf{L}_{m,n} \cong \mathbf{C}_m \times \mathbf{C}_n$ for every $m \geq n \geq 2$. This shows that all the m-by-n ladder posets are factorable.

(3) For some composite number c, let $c = n_1^{d_1} n_2^{d_2} \ldots n_r^{d_r}$, where $n_i, 1 \leq i \leq r$ are all distinct prime numbers and $d_i, 1 \leq i \leq r$ are positive integers. Then for every $r \geq 2$, we have $\langle D(c), | \rangle \cong \mathbf{C}_{d_1+1} \times \mathbf{C}_{d_2+1} \times \cdots \times \mathbf{C}_{d_r+1}$ $\cong \prod_{i=1}^{r} \mathbf{C}_{d_i+1}$. Thus, the poset of the divisors of a composite number c, where c has at least two distinct prime factors, is factorable.

2.4.6. Composite poset.

Definition 2.4.6 A poset $\mathbf{C}$ is said to be *composite* if it can be obtained from the ordinal product of two nonsingleton posets. Thus, the poset $\mathbf{C}$ is said to be *composite* if and only if their exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$.

Examples 2.4.6

(1) For any nonsingleton poset $\mathbf{A}$, we have $n\mathbf{A} \cong \mathbf{I}_n \otimes \mathbf{A}$ for all $n \geq 2$. This shows that every disconnected poset with pairwise isomorphic nonsingleton direct terms is trivially composite.
(2) The poset $2\mathbf{C}_2 \oplus \mathbf{C}_2$ is a nontrivial composite poset, because we have $2\mathbf{C}_2 \oplus \mathbf{C}_2 \cong \mathbf{B}_{2,1} \otimes \mathbf{C}_2$.
(3) Let $\mathbf{A}$ be a nonsingleton poset. In Section 3.6, we show by using the poset matrix that $\oplus^n \mathbf{A} \cong \mathbf{C}_n \otimes \mathbf{A}$ for every $n \geq 2$. This shows that all the posets $\oplus^n \mathbf{A}$, $n \geq 2$ are composite posets.

Remark 2.4.1 In Section 3.7, we show by using the poset matrix that every composite poset is decomposable. Conversely, we observe that the decomposable poset $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1} \cong \mathbf{C}_2[\mathbf{B}_{1,2}, \mathbf{B}_{2,1}]$ is not composite. Thus, a decomposable poset may not be a composite poset.

2.5. Recognition of posets

A frequently studied problem in the theory of posets is to recognize the classes of posets that satisfy some common structural properties. This problem is known as the Recognition Problem. One of the main aims of recognizing a particular class of posets is to specify how this class of posets is unique having its own identity in compared to other classes of posets. This process benefits dealing with a class of a big amount of structures by classifying them with a lesser amount of structures for many purposes. Since the number of posets increases exponentially with the number of elements in the posets, the recognition of various classes of posets was considered in the literature by numerous authors. In this section, we briefly discuss mainly four fundamental methods used repeatedly for the recognition of posets.

2.5.1. Set theoretic method.

In this method, the recognition of a particular class of posets is given by establishing some set theoretic concepts.

A poset $\mathbf{E} = \langle E, \leqslant \rangle$ is called an *interval order* if every $x, y \in E$ can be associated to the intervals $I_x = [a\ b]$ and $I_y = [c\ d]$ of real numbers such that

$$x \leqslant y \text{ if and only if either } I_x = I_y \text{ or } b < c.$$

For example, $\mathbf{B}_{2,2}$ is an interval order. To show this let $\mathbf{B}_{2,2} = \langle B, \leqslant \rangle$ such that $\min(B) = \{x_1, x_2\}$ and $\max(B) = \{x_3, x_4\}$. We associate x_1 to the interval $[0\ 2]$, x_2 to $[1\ 3]$, x_3 to $[4\ 6]$, and x_4 to $[5\ 7]$. Then, clearly, $x_1 \parallel x_2$ and $x_3 \parallel x_4$, and $x_i \prec x_j$ for all $x_i \in \min(B)$, $x_j \in \max(B)$.

For $x \in E$, the set $D(x) = \{y \in E : y \leqslant x \text{ and } x \neq y\}$ is called the *set of predecessors* of x. We define $D(E) = \{D(x) : x \in E\}$. For example, in the case of $\mathbf{B}_{2,2}$ as above, we have $D(x_1) = D(x_2) = \emptyset$, $D(x_3) = D(x_4) = \{x_1, x_2\}$, and hence $D(E) = \{\emptyset, \{x_1, x_2\}\}$.

Fishburn [**18**] gave a recognition of the interval order posets by using a set theoretic concept. He proved that a poset $\mathbf{E} = \langle E, \leqslant \rangle$ is an interval order if and only if the poset $\langle D(E), \subseteq \rangle$ is a chain.

2.5.2. Configuration method.

In this method, a class of posets is recognized by providing a minimum list of forbidden configurations.

Kaerkes [**27**] gave a recognition of series-parallel posets in terms of forbidden configuration. He showed that a poset is series-parallel if and only if it does not contain an induced subposet isomorphic to the N-shaped Hasse diagram, that is, the zigzag poset $\mathbf{Z_4}$.

Also, Fishburn [**18**] gave a recognition of interval order posets in terms of forbidden configuration. He showed that a poset is an interval order if and only if it does not contain an induced subposet isomorphic to the Hasse diagram having two 2-element parallel chains, that is, the poset $\mathbf{C}_2 + \mathbf{C}_2$ (Figure 2.17).

2.5.3. Matrix method.

This method includes associating common properties defined on an incidence matrix or precedence matrix or adjacency matrix.

Rhee [**46**] described a doubly-stochastic matrix D_P, an incidence matrix representation of a poset **P** and showed that if **P** is a series-parallel poset except a chain then D_P is singular, that is, $|D_P| = 0$.

In Section 4.4, we give a recognition of the factorable posets by using the poset matrix. We define the property of transitive blocks of poset matrices on a block poset matrix. Let the poset matrix M_n represents the poset **F**. We show that **F** is factorable if and only if M_n satisfies the property of transitive blocks of poset matrices.

2.5.4. Algorithmic method.

This method comprises a set of statements for listing, sorting, matching, and counting by generating all the nonisomorphic posets belong to the class of posets under consideration. Therefore, the acceptability of this method depends on the efficiency, that is, space and time complexities of the algorithm.

Khamis [**31**] described an algorithmic method to recognize if a finite poset is prime. For a poset **P**, he showed that there exists a polynomial-time algorithm that can determine, for every pair of distinct elements of P, if there exists a proper P-autonomous set consisting of those specified elements of P. This gives a recognition of prime posets. He applied this technique to give an enumeration of the class of prime posets.

2.6. Enumeration of posets

The fundamental task of the enumerative combinatorics is to count the number of elements of various finite collections. In the theory of posets, the problem of enumeration is to count the total number of n-element posets in a particular class of posets which we call the Enumeration Problem. Let an infinite collection of finite sets S_i, $i \in I$ where I is an index set, be given. Also let $f(i)$ be the number of elements in each S_i. We want to compute $f(i)$, $i \in I$ simultaneously, that is, we want to determine the cardinality of the set $\cup_{i \in I} S_i$. In most cases, immediate philosophical difficulties arise. Therefore, derivations of the counting function $f(i), i \in I$ is considered in literature in several standard ways. We discuss shortly some of these well-known methods in the rest of this section. For further details on this topic, please refer to the book by R. P. Stanley [**55**].

2.6.1. Exact enumeration.

This is the most satisfactory form of $f(i), i \in I$ involving only well-known functions. But in rare cases such a formula exists.

For example, in Section 5.3, we show by using the poset matrix that $CS(n)$, the number of n-element unlabeled connected P-series, can be expressed explicitly as follows:

$$CS(n) = 2^{n-1} - 1, \; n \geq 2$$

Also, in Section 5.4, we show that $CSP(n)$, the numbers of n-element unlabeled connected series-parallel posets, can be expressed as follows:

$$CSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} \prod_{i=1}^{m+1} DSP(r_{mji}), \; n \geq 2$$

Where $DSP(n)$ is the numbers of n-element unlabeled disconnected series-parallel posets. Here, the specific values of the parameters r_{mji}, for all $1 \leq m \leq n-1$, $1 \leq j \leq \binom{n-1}{m}$, and $1 \leq i \leq m+1$ are determined by an algorithm that runs in polynomial time.

2.6.2. Generating function.

This object is a formal power series. The two most common types of generating functions are *ordinary* generating function and *exponential* generating function.

It is shown by Stanly [**54**] that, $SP(n)$, the number of n-element unlabeled series-parallel posets, equals the coefficient of x^n in the generating function $F(x)$ given as follows:

$$F(x) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 48x^5 + 167x^6 + \cdots$$

2.6.3. Asymptotic estimation.

The asymptotic estimation of $f(i), i \in I$ frequently takes the form of $f(i) \sim g(i)$, where $g(i)$ is a familiar function for every $i \in I$. Sometimes the asymptotic estimates can be superior to some exact formulae that require lengthy computations.

For example, it is shown by Stanly [**54**] that $SP(n)$, that is, the coefficient of x^n in the above generating function $F(x)$ can be expressed asymptotically as follows:

$$SP(n) \sim C\ n^{-\frac{3}{2}}\ \alpha^{-n} \tag{1}$$

Here, C and α are some constants.

2.6.4. Algorithmic method.

This method comprises a set of statements for computing $f(i), i \in I$. Any counting function likely to arise in practice can be computed from an algorithm, so the acceptability of this method will depend on the elegance and performance of the algorithm.

Heitzig and Reinhold [**25**] were firstly able to give a general formula for counting the number of n-element unlabeled posets explicitly for $n \leq 14$. Here, they implemented an algorithm to generate the canonical posets and to count their automorphisms. Later on, Brinkmann and Mckay [**7**] described a more efficient algorithmic method to construct pairwise nonisomorphic posets by using the general formula given by Heitzig and Reinhold [**25**]. Here, they counted the automorphisms of the canonical posets by using the method given by Butler [**8**].

CHAPTER 3

Poset Matrix and its Associations to Posets

Due to many computational aspects, incidence matrices have classical applications in the recognition and enumeration of various classes of lattices, posets, graphs, and topologies. Fulkerson and Gross [**19**] characterized the interval graphs by using an incidence matrix of the dominant clique-vertex of graphs. Roberts [**48, 49**] characterized the proper interval graphs and Tucker [**57**] characterized the circular-arc graphs by using the augmented adjacency matrix of graphs. Skandera and Reed [**52**] described a connection between the f-vectors of the $(3 + 1)$-free posets and the unit-interval order by using anti-adjacency matrices. Rhee [**46**] described a doubly-stochastic matrix to represent posets and gave a matrix recognition of the class of series-parallel posets. These classical results motivated us to consider the notion of poset matrix, an incidence matrix to represent posets.

Butler [**8**] gave a particular solution to the problem of counting the number of n-element posets by interpreting a partial order relation on an n-element set as a nonsingular idempotent boolean relation matrix of order $n \geq 1$. This intuition gave us the idea of defining the properties of reflexivity, antisymmetry, and transitivity on a square $(0, 1)$-matrix which we call a poset matrix. We give the interpretations of the relabeling (simultaneous interchanges) of the rows and columns in a poset matrix. We recall some basic operations viz. the direct sum, ordinal sum, and Kronecker product of matrices and give the interpretations of these operations in the case of poset matrices. Further, we introduce the notion of the ordinal sum, ordinal product, and a composition of matrices and give the algebraic interpretations of these new operations in the case of poset matrices.

In Section 3.1, we introduce the notion of poset matrix and show how it can be associated to the posets. Also, we describe an interpretation regarding matrix transpose of a poset matrix.

In Section 3.2, we describe an interpretation regarding relabeling of poset matrices. We mainly show that every poset matrix can be relabeled into upper (equivalently, lower) triangular form with 1s in the main diagonal by a finite number of relabeling.

In Section 3.3, we recall the notion of the direct sum of matrices. We show that the direct sum of the poset matrices is also a poset matrix and it represents the direct sum of posets.

In Section 3.4, we introduce the notion of the ordinal sum of matrices. We show that the ordinal sum of the poset matrices is also a poset matrix and it represents the ordinal sum of posets.

In Section 3.5, we recall the well-known Kronecker product of matrices. We show that Kronecker product of poset matrices is also a poset matrix and it represents the direct product of posets.

In Section 3.6, we introduce the notion of the ordinal product of matrices. We mainly show that the ordinal product of poset matrices is also a poset matrix and it represents the ordinal product of posets. We also show that the ordinal product of poset matrices generalizes the ordinal sum of a collection of same poset matrices.

In Section 3.7, we introduce the notion of a composition of square matrices. We show that the composition of the poset matrices is also a poset matrix and it represents the composition of posets. Then we show that the composition of the poset matrices gives a generalization of the ordinal product of poset matrices. We also show that the class of decomposable posets generalizes the classes of composite posets and series-parallel posets.

From now on we use the notations $M_{m,n}$ for an m-by-n matrix and M_m for a square matrix of order m. In particular, we use I_n for the identity matrix of order n and C_n for the matrix $[c_{ij}], 1 \leq i, j \leq n$ defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise. Note that $C_1 = I_1 = 1$.

3.1. The notion of poset matrix

Definition 3.1.1 A square $(0,1)$-matrix $M = [a_{ij}], 1 \leq i, j \leq n$ is called a *poset matrix* if the following conditions hold.

(1) $a_{ii} = 1$ for all $1 \leq i \leq n$ i.e. M is *reflexive*;

(2) $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ i.e. M is *antisymmetric*;

(3) $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$ i.e. M is *transitive*.

Example 3.1.1

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the above example, all the square $(0,1)$-matrices I_2, C_3, and N are both reflexive and antisymmetric, because these are upper triangular and have all entries 1s in the main diagonal. Also, all of these are trivially transitive. Therefore, the matrices I_2, C_3, and N are all poset matrices.

On the other hand, none of the square $(0,1)$-matrices $P = [p_{ij}]$, $1 \leq i, j \leq 2$, $Q = [q_{ij}]$, $1 \leq i, j \leq 3$, and $R = [r_{ij}]$, $1 \leq i, j \leq 4$, as in the following example, is a poset matrix. Because,

(1) $p_{22} = 0$ implies P is not reflexive,

(2) $q_{13} = q_{31} = 1$ implies Q is not antisymmetric, and

(3) $r_{12} = r_{24} = 1$ but $r_{14} \neq 1$ imply R is not transitive.

Example 3.1.2

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is clearly reflexive and antisymmetric. Therefore, an upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is a poset matrix if it is transitive. For every $n \geq 1$, both the matrices I_n and C_n are poset matrices because these are upper triangular with 1s in the main diagonal and trivially transitive.

To each poset matrix we can associate a poset. Suppose that $M_m = [a_{ij}]$, $1 \leq i,j \leq m$ is a poset matrix and $P = \{x_1, x_2, \ldots, x_m\}$, where x_i corresponds the i-th row (or column) of M_m. We define a relation $\leqslant$ on P such that for all $1 \leq i,j \leq m$,

$$x_i \leqslant x_j \text{ if and only if } a_{ij} = 1.$$

Since M_m is a poset matrix, clearly $\leqslant$ is an order relation on P. Thus $\mathbf{P} = \langle P, \leqslant \rangle$ is a poset. We say that the poset matrix M_m *represents* the poset $\mathbf{P}$ and vice versa.

For example, the poset matrices I_2, C_3, and N, as in the Example 3.1.1, represent the posets $\mathbf{I}_2$, $\mathbf{C}_3$, and $\mathbf{Z}_4$, as shown in Figure 3.1 below, respectively. Also, the poset matrices I_n and C_n represent the posets $\mathbf{I}_n$, where $x_i || x_j$ for all $1 \leq i,j \leq n$, and $\mathbf{C}_n$, where $x_1 \prec x_2 \prec \cdots \prec x_n$, both with the same ground set $\{x_1, x_2, \ldots, x_n\}$, respectively. In particular, both I_1 and C_1 represent the singleton poset $\mathbf{1}$.

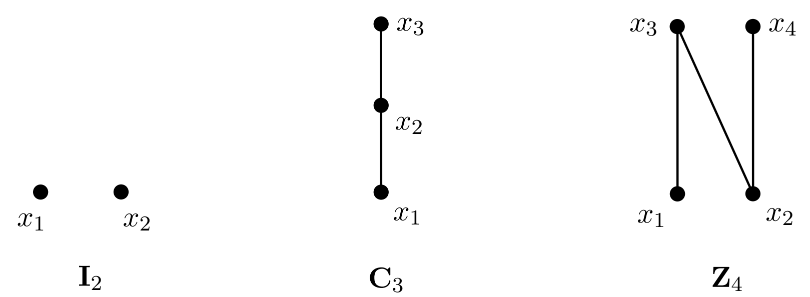


FIGURE 3.1. Hasse diagrams of $\mathbf{I}_2$, $\mathbf{C}_3$, and $\mathbf{Z}_4$ with labeling.

Consider the following example where L^t equals the matrix transpose of the matrix L.

Example 3.1.3

$$L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad\qquad L^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Here, both the matrices L and L^t are poset matrices. We observe that L represents the poset $\mathbf{B}_{2,1}$ and L^t represents the poset $\mathbf{B}_{1,2}$ (Figure 3.2). Clearly, $\mathbf{B}_{1,2}$ is dual to the poset $\mathbf{B}_{2,1}$. We establish this result in general in the following.

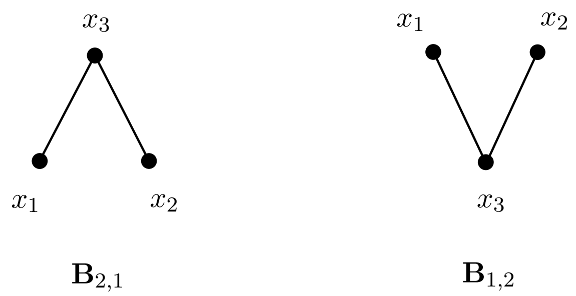


FIGURE 3.2. Hasse diagrams of $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2}$ with labeling.

Theorem 3.1.1 The matrix transpose M^t of a poset matrix M is a poset matrix and it represents the poset dual to the poset represented by M.

Proof. Let $M = [a_{ij}], 1 \leq i, j \leq n$. Then $M^t = [a_{ji}], 1 \leq i, j \leq n$. Since $a_{ii} = 1$ for all $1 \leq i \leq n$ in both M and M^t, clearly M^t is reflexive. Also $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ in both M and M^t. Thus M^t is antisymmetric. Let $a_{ij} = 1$ and $a_{jk} = 1$ in M^t. Then $a_{ji} = 1$ and $a_{kj} = 1$ in M. Since M is transitive, $a_{ki} = 1$ in M. This implies $a_{ik} = 1$ in M^t. Thus M^t is transitive. Therefore, M^t is a poset matrix.

To show that M^t represents the poset dual to the poset represented by M, let M represents the poset $\mathbf{P} = \langle X, \leqslant_P \rangle$ and M^t represents the poset $\mathbf{Q} = \langle X, \leqslant_Q \rangle$ with the same underlying set $X = \{x_1, x_2, \ldots, x_n\}$. Let $x_i \leqslant_P x_j$ for some i and j. Then $a_{ij} = 1$ in M implies $a_{ji} = 1$ in M^t. Thus $x_j \leqslant_Q x_i$. This shows that the order $\leqslant_Q$ is dual to $\leqslant_P$ on X, that is, $\mathbf{Q} \cong \mathbf{P}^\partial$. ∎

3.2. Relabeling of the poset matrix

Definition 3.2.1 Let M_m be a poset matrix. Then for some $1 \leq i, j \leq m$, interchanges of i-th and j-th rows along with interchanges of i-th and j-th columns in M_m is called *(i,j)-relabeling* of M_m.

Example 3.2.1

$$L^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{(1,3)-relabeling}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L^{'}$$

The above example shows a relabeling of the poset matrix L^t (Example 3.1.3). Here, we firstly interchange the rows 1 and 3 of L^t and then interchange the columns 1 and 3 of L^t, and finally obtain the matrix $L^{'}$. Below we give another example of relabeling.

Example 3.2.2

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{(1,2)-relabeling}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{(2,3)-relabeling}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{(1,2)-relabeling}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M^{'}$$

We observe that the matrix $M^{'}$, as in the above example, obtained by some relabelings of the poset matrix M, equals the poset matrix C_4. Moreover, M and $M^{'}$ represent the same poset $\mathbf{C}_4$ with the ground set $\{x_1, x_2, x_3, x_4\}$ such that $x_3 \prec x_2 \prec x_1 \prec x_4$ on M and $x_1 \prec x_2 \prec x_3 \prec x_4$ on $M^{'}$. Thus relabeling of a poset matrix M does not make any change to the order relation on M, it provides just a renaming of the elements in the poset represented by M. We establish these facts in the following.

Theorem 3.2.1 Any relabeling of a poset matrix is a poset matrix and it represents the same poset up to isomorphism.

Proof. Let $M_m = [a_{ij}]$ be a poset matrix and $N_m = [b_{ij}]$ be the matrix obtained by (k,l)-relabeling of M_m. Then we have the following equalities.

(1) $b_{kk} = a_{ll}$ and $b_{ll} = a_{kk}$

(2) $b_{kl} = a_{lk}$ and $b_{lk} = a_{kl}$

(3) $b_{ik} = a_{il}$ and $b_{il} = a_{ik}$ for all $1 \leq i \leq m, i \neq k, i \neq l$

(4) $b_{kj} = a_{lj}$ and $b_{lj} = a_{kj}$ for all $1 \leq j \leq m, j \neq k, j \neq l$

(5) $b_{ij} = a_{ij}$ for all $1 \leq i, j \leq m, i \neq k, i \neq l, j \neq k, j \neq l$

Since M_m is a poset matrix, the above equalities show that N_m is a poset matrix.

To show that $M_m = [a_{ij}]$ and $N_m = [b_{ij}]$ represent the same poset up to isomorphism, let they represent $\mathbf{P} = \langle P, \leqslant_P \rangle$ and $\mathbf{Q} = \langle Q, \leqslant_Q \rangle$, respectively, where $P = \{x_1, x_2, \ldots, x_m\}$ and $Q = \{y_1, y_2, \ldots, y_m\}$. Define $\phi : P \to Q$ as follows.

$$\phi(x_t) = \begin{cases} y_l \text{ if } t = k, \\ y_k \text{ if } t = l, \\ y_t \text{ otherwise.} \end{cases}$$

Since $|P| = m = |Q|$, clearly ϕ is bijective. Let $x_i \leqslant_P x_j$. Then we have the following cases.

(1) $i = k$.

(a) $j = k$. Then $1 = a_{ij} = a_{kk} = b_{ll}$ (equality 1) implies $y_l \leqslant_Q y_l$.

(b) $j = l$. Then $1 = a_{ij} = a_{kl} = b_{lk}$ (equality 2) implies $y_l \leqslant_Q y_k$.

(c) $j \neq k, j \neq l$. Then $1 = a_{ij} = a_{kj} = b_{lj}$ (equality 4) implies $y_l \leqslant_Q y_j$.

(2) $i = l$.

(a) $j = k$. Then $1 = a_{ij} = a_{lk} = b_{kl}$ (equality 2) implies $y_k \leqslant_Q y_l$.

(b) $j = l$. Then $1 = a_{ij} = a_{ll} = b_{kk}$ (equality 1) implies $y_k \leqslant_Q y_k$.

(c) $j \neq k, j \neq l$. Then $1 = a_{ij} = a_{lj} = b_{kj}$ (equality 4) implies $y_k \leqslant_Q y_j$.

(3) $i \neq k, i \neq l$.

(a) $j = k$. Then $1 = a_{ij} = a_{ik} = b_{il}$ (equality 3) implies $y_i \leqslant_Q y_l$.

(b) $j = l$. Then $1 = a_{ij} = a_{il} = b_{ik}$ (equality 3) implies $y_i \leqslant_Q y_k$.

(c) $j \neq k, j \neq l$. Then $1 = a_{ij} = b_{ij}$ (equality 5) implies $y_i \leqslant_Q y_j$.

Thus $\phi(x_i) \leqslant_Q \phi(x_j)$ and $\phi : \mathbf{P} \to \mathbf{Q}$ is an order isomorphism. ∎

We observe that the poset matrix $M^{'}$ (Example 3.2.2) obtained by some relabeling of the non-triangular poset matrix M is in upper triangular form. Below we establish this result in general.

Theorem 3.2.2 Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite times of relabeling.

Proof. Let $M = [a_{ij}], 1 \leq i, j \leq n$ be a poset matrix. Suppose $a_{ij} = 1$ for some $i > j$ such that for all $1 \leq k \leq j-1$, $a_{ik} = 0$ and the partition $[a_{uv}], 1 \leq u, v \leq i-1$ of M is upper triangular. Then $a_{(i-1)i} = 0$. Otherwise, since $a_{ij} = 1$, we have the following cases.

(1) $i = j + 1$. Then $a_{ji} = a_{(i-1)i}$ contradicts that M is antisymmetric.

(2) $i > j + 1$. Then transitivity of M implies $a_{(i-1)j} = 1$ which contradicts that P is upper triangular.

Then by (i, j)-relabeling of M, we have $a_{(i-1)j} = 1$ and $a_{ik} = 0$ for all $1 \leq k \leq i - 1$. Similarly, the matrix $M^{'}$ obtained by $(k, k-1)$-relabeling of M for all $i - 1 \geq k \geq j + 1$ with $i > j + 1$ is a matrix having partition $[a_{uv}]$, $1 \leq u, v \leq i$ of $M^{'}$ in upper triangular form. Since n is finite, continuing the same process for all $1 \leq j < i \leq n$ with $a_{ij} = 1$, we have $M^{'}$ in upper triangular form. By Theorem 3.2.1, $M^{'}$ is a poset matrix and hence it has 1s in the main diagonal. We show similarly the lower triangular case. ∎

Corollary 3.2.3 Let M be any square $(0, 1)$-matrix. Then M is a poset matrix if and only if M is transitive and upper (or lower) triangular with 1s in the main diagonal.

Notes 3.2.1 From now on by a poset matrix we mean a poset matrix in upper triangular form. Also, we use the notations $Z_{m,n}$ for an m-by-n matrix having entries 0s only and $O_{m,n}$ for an m-by-n matrix having entries 1s only.

3.3. Direct sum of poset matrices

Definition 3.3.1 The *direct sum* of the matrices $M_{m,p}$ and $N_{n,q}$, denoted by $M_{m,p} \oplus N_{n,q}$, is defined as an $(m+n)$-by-$(p+q)$ block matrix such that

$$M_{m,p} \oplus N_{n,q} = \begin{bmatrix} M_{m,p} & | & Z_{m,q} \\ -- & . & -- \\ Z_{n,p} & | & N_{n,q} \end{bmatrix}$$

In this case, we call $M_{m,p}$ and $N_{n,q}$ as the *direct terms* of $M_{m,p} \oplus N_{n,q}$.

The following example shows two direct sums of the poset matrices L and $L^{'}$ given in Example 3.1.3 and Example 3.2.1.

Example 3.3.1

$$L \oplus L^{'} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & | & 0 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 \\ - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$L^{'} \oplus L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 \\ - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

In general, $S_s = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$, where $s = \sum_{i=1}^{n} m_i$, can be given as follows:

$$S_s = \begin{bmatrix} M_{m_1} & Z_{m_1,m_2} & \cdots & Z_{m_1,m_n} \\ Z_{m_2,m_1} & M_{m_2} & \cdots & Z_{m_2,m_n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{m_n,m_1} & Z_{m_n,m_2} & \cdots & M_{m_n} \end{bmatrix}$$

Here, the blocks of $S_s = [S_{ij}]$, $1 \le i, j \le n$, where $s = \sum_{i=1}^{n} m_i$, can be explained as follows:

$$S_{ij} = \begin{cases} M_{m_i} \text{ if } i = j, \\ Z_{m_i,m_j} \text{ if } i < j, \\ Z_{m_j,m_i} \text{ otherwise.} \end{cases}$$

We observe that both the block matrices $L \oplus L'$ and $L' \oplus L$, as given in the above example, are poset matrices and represent the posets $\mathbf{B}_{2,1} + \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$, respectively. These can be checked immediately from the labeled Hasse diagrams shown in Figure 3.3. We establish this fact in the following which gives an association of the direct sum of poset matrices to posets.

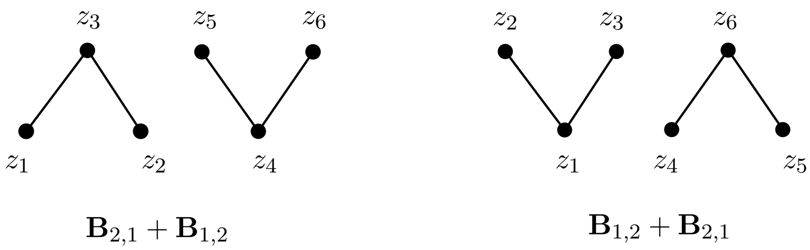


FIGURE 3.3. Hasse diagrams of $\mathbf{B}_{2,1} + \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$ with labeling.

Theorem 3.3.1 Let M_m represents the poset $\mathbf{A}$ and N_n represents the poset $\mathbf{B}$. Then the matrix $M_m \oplus N_n$ is a poset matrix and it represents the poset $\mathbf{A}+\mathbf{B}$.

Proof. Let $M_m = [a_{ij}]$, $N_n = [b_{ij}]$ and $M_m \oplus N_n = S_{m+n} = [s_{ij}]$ with block representation $[S_{ij}], 1 \le i, j \le 2$. Since S_{m+n} is upper triangular with elements 1s in the main diagonal, because $S_{21} = Z_{n,m}$, and M_m and N_n are poset matrices, S_{m+n} is clearly reflexive and antisymmetric. For transitivity of S_{m+n}, let $s_{ij} = s_{jk} = 1$ for some $i \le j \le k$. Then we have the following cases.

(1) $k \le m$. Then $i \le j \le k \le m$ implies $s_{ij}, s_{jk}, s_{ik} \in M_m$. Since M_m is transitive, s_{ik}=1.

(2) $k > m$.

   (a) $j \le m$. Then $j \le m < k$ implies $s_{jk} \in Z_{m,n}$ which contradicts that $s_{jk} = 1$.

(b) $j > m$.

(i) $i \leq m$. Then $i \leq m < j$ implies $s_{ij} \in Z_{m,n}$ which contradicts that $s_{ij} = 1$.

(ii) $i > m$. Then $m < i \leq j \leq k$ implies $s_{ij}, s_{jk}, s_{ik} \in N_n$. Since N_n is transitive, s_{ik}=1.

Thus S_{m+n} is transitive and hence a poset matrix.

Let $\mathbf{A} = \langle A; \leqslant_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We show that S_{m+n} represents the poset $\mathbf{A} + \mathbf{B} = \langle A \cup B; \leqslant_+ \rangle$, where $A \cup B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $s_{ij} = 1$ in S_{m+n} for some $1 \leq i, j \leq m+n$. Then either $s_{ij} \in M_m$ or $s_{ij} \in N_n$. Since M_m and N_n represent $\mathbf{A}$ and $\mathbf{B}$, respectively, either $x_i \leqslant_A x_j$ or $x_i \leqslant_B x_j$. By definition, $x_i \leqslant_+ x_j$ in $A \cup B$. Hence S_{m+n} represents the poset $\mathbf{A} + \mathbf{B}$. ∎

The following theorem gives a generalization of Theorem 3.3.1.

Theorem 3.3.2 Let $M_{m_i}, 1 \leq i \leq n$ be the poset matrices representing the posets $\mathbf{P}_i, 1 \leq i \leq n$, respectively. Then $M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ is a poset matrix and it represents the poset $\sum_{i=1}^{n} \mathbf{P}_i$.

Proof. The proof follows inductively by Theorem 3.3.1. ∎

3.4. Ordinal sum of poset matrices

Definition 3.4.1 The *ordinal sum* of the matrices $M_{m,p}$ and $N_{n,q}$, denoted by $M_{m,p} \boxplus N_{n,q}$, is defined as an $(m+n)$-by-$(p+q)$ block matrix such that

$$M_{m,p} \boxplus N_{n,q} = \begin{bmatrix} M_{m,p} & | & O_{m,q} \\ -- & . & -- \\ Z_{n,p} & | & N_{n,q} \end{bmatrix}$$

In this case, we call $M_{m,p}$ and $N_{n,q}$ as the *ordinal terms* of $M_{m,p} \boxplus N_{n,q}$.

The following example shows two ordinal sums of the poset matrices L and L' given in Example 3.1.3 and Example 3.2.1.

Example 3.4.1

$$L \boxplus L' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & | & 1 & 1 & 1 \\ 0 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 0 & 1 & | & 1 & 1 & 1 \\ - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$L' \boxplus L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 1 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 1 & | & 1 & 1 & 1 \\ - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

In general, $T_t = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$, where $t = \sum_{i=1}^{n} m_i$, can be given as follows:

$$T_t = \begin{bmatrix} M_{m_1} & O_{m_1,m_2} & \cdots & O_{m_1,m_n} \\ Z_{m_2,m_1} & M_{m_2} & \cdots & O_{m_2,m_n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{m_n,m_1} & Z_{m_n,m_2} & \cdots & M_{m_n} \end{bmatrix}$$

Here, the blocks of $T_t = [T_{ij}]$, $1 \leq i,j \leq n$, where $t = \sum_{i=1}^{n} m_i$, can be explained as follows:

$$T_{ij} = \begin{cases} M_{m_i} \text{ if } i = j, \\ O_{m_i,m_j} \text{ if } i < j, \\ Z_{m_j,m_i} \text{ otherwise.} \end{cases}$$

We observe that both the block matrices $L \boxplus L'$ and $L' \boxplus L$, as given in the above example, are poset matrices and represent the posets $\mathbf{B}_{2,1} \oplus \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$, respectively. These can be checked immediately from the labeled Hasse diagrams shown in Figure 3.4. We establish this fact in the following which gives an association of the direct sum of poset matrices to posets.

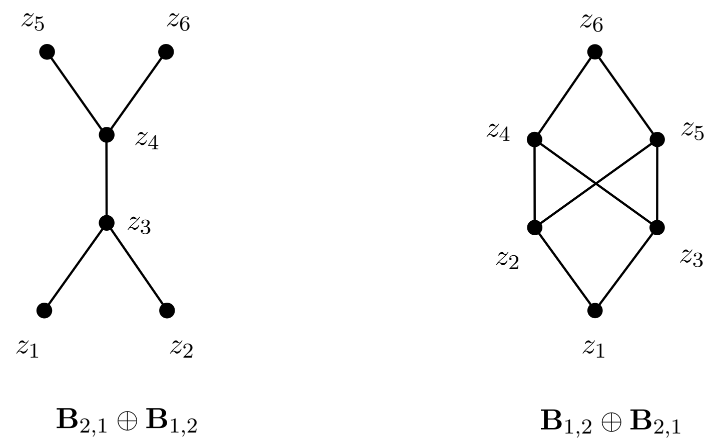


FIGURE 3.4. Hasse diagrams of $\mathbf{B}_{2,1} \oplus \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ with labeling.

Theorem 3.4.1 Let M_m represents the poset $\mathbf{A}$ and N_n represents the poset $\mathbf{B}$. Then the matrix $M_m \boxplus N_n$ is a poset matrix and it represents the poset $\mathbf{A} \oplus \mathbf{B}$.

Proof. Let $M_m = [a_{ij}]$, $N_n = [b_{ij}]$ and $M_m \boxplus N_n = T_{m+n} = [t_{ij}]$ with block representation $[T_{ij}], 1 \leq i, j \leq 2$. Since T_{m+n} is upper triangular with elements 1s in the main diagonal, because $T_{21} = Z_{n,m}$, and M_m and N_n are poset matrices, T_{m+n} is clearly reflexive and antisymmetric. For transitivity of T_{m+n}, let $t_{ij} = t_{jk} = 1$ for some $i \leq j \leq k$. Then we have the following cases.

(1) $k \leq m$. Then $i \leq j \leq k \leq m$ implies $t_{ij}, t_{jk}, t_{ik} \in M_m$. Since M_m is transitive, t_{ik}=1.

(2) $k > m$.
   (a) $j \leq m$. Then $i \leq j \leq m < k$ implies $t_{ik} \in O_{m,n}$ and hence t_{ik}=1.
   (b) $j > m$.
      (i) $i \leq m$. Then $i \leq m < j \leq k$ implies $t_{ik} \in O_{m,n}$ and hence t_{ik}=1.
      (ii) $m < i$. Then $m < i \leq j \leq k$ implies $t_{ij}, t_{jk}, t_{ik} \in N_n$ and since N_n is transitive, t_{ik}=1.

Thus T_{m+n} is transitive and hence a poset matrix.

Let $\mathbf{A} = \langle A; \leqslant_A \rangle$, where $A = \{x_1,\ x_2,\ \ldots,\ x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We show that T_{m+n} represents the poset $\mathbf{A} \oplus \mathbf{B} = \langle A \cup B; \leqslant_\oplus \rangle$, where $A \cup B = \{x_1,\ x_2,\ \ldots,\ x_m,\ x_{m+1},\ x_{m+2},\ \ldots,\ x_{m+n}\}$. Let $t_{ij} = 1$ in T_{m+n} for some $1 \leq i, j \leq m+n$. Then either $t_{ij} \in M_m$ or $t_{ij} \in N_n$ or $t_{ij} \in O_{m,n}$. Therefore, either $x_i \leqslant_A x_j$ (because M_m represents $\mathbf{A}$) or $x_i \leqslant_B x_j$ (because N_n represents $\mathbf{B}$) or $x_i \in A$ and $x_j \in B$. By definition, $x_i \leqslant_\oplus x_j$ in $A \cup B$. Hence T_{m+n} represents $\mathbf{A} \oplus \mathbf{B}$. ∎

The following theorem gives a generalization of Theorem 3.4.1.

Theorem 3.4.2 Let $M_{m_i}, 1 \leq i \leq n$ be the poset matrices representing the posets $\mathbf{P}_i, 1 \leq i \leq n$, respectively. Then $M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$ is a poset matrix and it represents the poset $\bigoplus_{i=1}^{n} \mathbf{P}_i$.

Proof. The proof follows inductively by Theorem 3.4.1. ∎

3.5. Kronecker product of poset matrices

According to Van Loan [**59**], the application areas where Kronecker products are abound are all thriving which include particularly the areas of signal processing, image processing, semidefinite programming, and quantum computing. The underlying fact is that the Kronecker product has a rich and very pleasing algebra supporting a wide range of fast, elegant, and practical algorithms. Therefore, Kronecker product of various matrices are considered by numerous authors and shown their applications to the related fields [**34, 59, 60**].

Definition 3.5.1 The *Kronecker product* (or *tensor product* or *direct product*) of the matrices $M_{m,n} = [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$ and $N_{p,q}$, denoted by $M_{m,n} \otimes N_{p,q}$, is defined as an $(m \times p)$-by-$(n \times q)$ block matrix such that

$$M_{m,n} \otimes N_{p,q} = \begin{bmatrix} a_{11}N_{p,q} & a_{12}N_{p,q} & \cdots & a_{1n}N_{p,q} \\ a_{21}N_{p,q} & a_{22}N_{p,q} & \cdots & a_{2n}N_{p,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}N_{p,q} & a_{m1}N_{p,q} & \cdots & a_{mn}N_{p,q} \end{bmatrix}$$

Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and N_n be poset matrices. Since M_m is a (0,1)-matrix, the (i, j)-th block P_{ij} of $P_{m\times n} = M_m \otimes N_n = [P_{ij}]$, $1 \leq i, j \leq m$ can be expressed as follows:

$$P_{ij} = \begin{cases} N_n \text{ if } a_{ij} = 1, \\ Z_n \text{ otherwise.} \end{cases} \tag{2}$$

The following example shows the Kronecker product of the poset matrices L and $L^{'}$ given in Example 3.1.3 and Example 3.2.1, respectively.

Example 3.5.1

$$L \otimes L^{'} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We observe that the block matrix $L \otimes L^{'}$ is a poset matrix and represents the poset $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ that can be checked immediately from the Hasse diagram shown in the Figure 3.5. We establish this result in the following which gives an association of the Kronecker product of poset matrices to posets.

Theorem 3.5.1 Let the poset matrix M_m represents the poset $\mathbf{A}$ and the poset matrix N_n represents the poset $\mathbf{B}$. Then the matrix $M_m \otimes N_n$ is a poset matrix and it represents the poset $\mathbf{A} \times \mathbf{B}$.

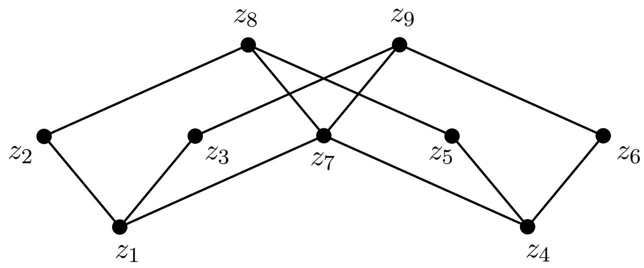


FIGURE 3.5. Hasse diagram of $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ with labeling.

Proof. Let $M_m = [a_{ij}], 1 \le i, j \le m$ and $N_n = [b_{ij}], 1 \le i, j \le n$. Also let $\mathbf{A} = \langle A; \leqslant_A \rangle$ where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$ where $B = \{y_1, y_2, \ldots, y_n\}$. Also let $M_m \otimes N_n = P_{m\times n} = [p_{ij}], 1 \le i, j \le m \times n$ with block representation $[P_{ij}], 1 \le i, j \le m$. Since both M_m and N_n are upper triangular matrices, $P_{ij} = Z_n$ for all $i > j$. Thus $P_{m\times n}$ is upper triangular with elements 1s in the main diagonal and hence $P_{m\times n}$ is clearly reflexive and antisymmetric. For transitivity of $P_{m\times n}$, let $p_{ij} = p_{jk} = 1$ for some $1 \le i \le j \le k \le m \times n$. Then we have the following three cases:

(1) $p_{ij}, p_{jk} \in P_{rr} = N_n$ for some $1 \le r \le m$. Then there exist $b_{i'j'}, b_{j'k'}, b_{i'k'} \in N_n$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since N_n is transitive, $q_{ik} = b_{i'k'} = 1$.

(2) $p_{ij} \in P_{rs} = N_n$ and $p_{jk} \in P_{ss} = N_n$ for some $1 \le r < s \le m$. Then $p_{ik} \in P_{rs} = N_n$ and hence $p_{ik} = 1$.

(3) $p_{ij} \in P_{rs} = N_n$ and $p_{jk} \in P_{st} = N_n$ for some $1 \le r < s < t \le m$. Then $p_{ik} \in P_{rt}$. Then, by the definition of Kronecker product of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since M_m is transitive, $a_{rt} = 1$. Therefore $P_{rt} = N_n$ and, clearly, $p_{ik} = 1$.

Thus $P_{m\times n}$ is transitive and hence a poset matrix.

Now we show that $P_{m\times n}$ represents $\mathbf{A} \times \mathbf{B} = \langle A \times B; \leqslant_\times \rangle$, where the ground set $A \times B = \{(x_k, y_r) : 1 \le k \le m, 1 \le r \le n\}$. Then we have $A \times B \cong \{z_i : 1 \le i \le m \times n\} = Z$, because, the mapping $(x_k, y_r) \mapsto z_i$ such that $n(k-1) + r = i$ gives an one-to-one correspondence between $A \times B$ and Z. Let $p_{ij} = 1$ in $P_{m\times n}$ for some $1 \le i \le j \le m \times n$. Assign $r = i \mod n$, $s = j \mod n$, $k = \frac{i-r}{n} + 1$

and $l = \frac{j-s}{n} + 1$. Then $z_i \mapsto (x_k, y_r)$, $z_j \mapsto (x_l, y_s)$ and $p_{ij} = b_{kl} \in Q_{rs} = N_n$. Thus $b_{kl} = 1$ in N_n and, by the definition of Kronecker product of poset matrices, $a_{rs} = 1$ in M_m. Since N_n represents $\mathbf{A}$ and M_m represents $\mathbf{B}$, $x_k \leqslant_A x_l$ and $y_r \leqslant_B y_s$. Then, by the definition of direct product of posets, $(x_k, y_r) \leqslant_\times (x_l, y_s)$ i.e. $z_i \leqslant_\times z_j$. For the converse, similarly, we show that $z_i \leqslant_\times z_j$ for some $1 \leq i, j \leq m \times n$ implies $p_{ij} = 1$ in $P_{m\times n}$. Hence $P_{m\times n}$ represents $\mathbf{A} \times \mathbf{B}$. ∎

3.6. Ordinal product of poset matrices

Definition 3.6.1 The *ordinal product* of the m-by-n matrix $M_{m,n} = [a_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$ and the p-by-q matrix $N_{p,q}$, denoted by $M_{m,n} \boxtimes N_{p,q}$, is defined as an $(m \times p)$-by-$(n \times q)$ block matrix such that

$$M_{m,n} \boxtimes N_{p,q} = \begin{bmatrix} a_{11}N_{p,q} & a_{12}O_{p,q} & \cdots & a_{1n}O_{p,q} \\ a_{21}O_{p,q} & a_{22}N_{p,q} & \cdots & a_{2n}O_{p,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}O_{p,q} & a_{m1}O_{p,q} & \cdots & a_{mn}N_{p,q} \end{bmatrix}$$

Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and N_n be poset matrices. Since M_m is a (0,1)-matrix, the (i, j)-th block Q_{ij} of the matrix $Q_{m\times n} = M_m \boxtimes N_n = [Q_{ij}]$, $1 \leq i, j \leq m$ can be expressed as follows:

$$Q_{ij} = \begin{cases} N_n \text{ if } i = j, \\ O_n \text{ if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n \text{ otherwise.} \end{cases} \tag{3}$$

The following example shows the ordinal product of the poset matrices L and $L^{'}$ given in Example 3.1.3 and Example 3.2.1, respectively.

Example 3.6.1

$$L \boxtimes L' = \begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 1 & 0 & | & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 1 & | & 0 & 0 & 0 & | & 1 & 1 & 1 \\ - & - & - & . & - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 1 & | & 1 & 1 & 1 \\ - & - & - & . & - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

We observe that the block matrix $L \boxtimes L^{'}$ is a poset matrix and it represents the poset $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ that can be checked immediately from the Hasse diagram shown in the Figure 3.6. We establish this result in the following which gives an association of the ordinal product of poset matrices to posets.

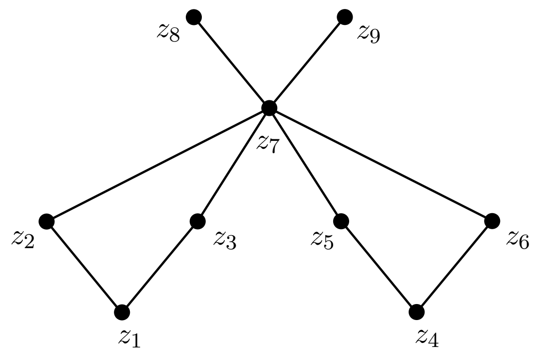


FIGURE 3.6. Hasse diagram of $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ with labeling.

Theorem 3.6.1 Let the poset matrix M_m represents the poset $\mathbf{A}$ and the poset matrix N_n represents the poset $\mathbf{B}$. Then the matrix $M_m \boxtimes N_n$ is a poset matrix and it represents the poset $\mathbf{A} \otimes \mathbf{B}$.

Proof. Let $M_m = [a_{ij}], 1 \leq i,j \leq m$ and $N_n = [b_{ij}], 1 \leq i,j \leq n$. Also, let $\mathbf{A} = \langle A; \leqslant_A \rangle$ where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$ where $B = \{y_1, y_2, \ldots, y_n\}$. Also let $M_m \boxtimes N_n = Q_{m \times n} = [q_{ij}], 1 \leq i,j \leq m \times n$ with block representation $[Q_{ij}], 1 \leq i, j \leq m$. Since both M_m and N_n are upper triangular matrices, $Q_{ij} = Z_n$ for all $i > j$. Thus $Q_{m\times n}$ is upper triangular with elements 1s in the main diagonal and hence $Q_{m\times n}$ is clearly reflexive and antisymmetric. For transitivity of $Q_{m\times n}$, let $q_{ij} = q_{jk} = 1$ for some $1 \leq i \leq j \leq k \leq m \times n$. Then we have the following three cases:

(1) $q_{ij}, q_{jk} \in Q_{rr} = N_n$ for some $1 \leq r \leq m$. Then there exist $b_{i'j'}, b_{j'k'}, b_{i'k'} \in N_n$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since N_n is transitive, $q_{ik} = b_{i'k'} = 1$.

(2) $q_{ij} \in Q_{rs} = O_n$ and $q_{jk} \in Q_{ss} = N_n$ for some $1 \leq r < s \leq m$. Then $q_{ik} \in Q_{rs} = O_n$ and clearly $q_{ik} = 1$.

(3) $q_{ij} \in Q_{rs} = O_n$ and $q_{jk} \in Q_{st} = O_n$ for some $1 \leq r < s < t \leq m$. Then $q_{ik} \in Q_{rt}$. Then, by the definition of ordinal product of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since M_m is transitive, $a_{rt} = 1$. Therefore $Q_{rt} = O_n$ and clearly $q_{ik} = 1$.

Thus $Q_{m\times n}$ is transitive and hence a poset matrix.

Now we show that $Q_{m\times n}$ represents $\mathbf{A} \otimes \mathbf{B} = \langle A \times B; \leqslant_\otimes \rangle$, where $A \times B = \{(x_k, y_r) : 1 \leq k \leq m, 1 \leq r \leq n\}$. Since the mapping $(x_k, y_r) \mapsto z_i$, where $n(k-1)+r = i$, gives an one-to-one correspondence, $A \times B \cong \{z_i : 1 \leq i \leq m \times n\}$. Let $q_{ij} = 1$ in $Q_{m\times n}$ for some $1 \leq i \leq j \leq m \times n$. Assign $r = i \mod n$, $s = j \mod n$, $k = \frac{i-r}{n} + 1$ and $l = \frac{j-s}{n} + 1$. Then $(x_k, y_r) \mapsto z_i$, $(x_l, y_s) \mapsto z_j$ and we have the following cases:

(1) $k = l$. Then $Q_{kl} = N_n$ and $b_{rs} = q_{ij} \in Q_{kl} = N_n$. Then $x_k = x_l$ in A and, since N_n represents $\mathbf{B}$, $y_r \leqslant_B y_l$. Then by the definition of ordinal product of posets, $(x_k, y_r) \leqslant_\otimes (x_l, y_s)$ i.e. $z_i \leqslant_\otimes z_j$.

(2) $k < l$. Then $Q_{kl} = O_n$. By the definition of ordinal product of poset matrix, $a_{kl} \in M_m$ and $a_{kl} = 1$. Since M_m represents $\mathbf{A}$, $x_k \leqslant_A x_l$. Then, by the definition of ordinal product of posets, $(x_k, y_r) \leqslant_\otimes (x_l, y_s)$ i.e. $z_i \leqslant_\otimes z_j$.

For the converse, similarly, we show that $z_i \leqslant_\otimes z_j$ for some $1 \leq i, j \leq m \times n$ implies $q_{ij} = 1$ in $Q_{m\times n}$. Hence $Q_{m\times n}$ represents $\mathbf{A} \otimes \mathbf{B}$. ∎

Proposition 3.6.2 Let $\mathbf{B}$ be any poset. Then $\mathbf{C}_m \otimes \mathbf{B} \cong \oplus^m \mathbf{B}$.

Proof. Let the poset matrix N_n represents the poset $\mathbf{B}$. We first show that $C_m \boxtimes N_n = \boxplus^m N_n$. By Theorem 3.6.1 and Theorem 3.4.2, both $C_m \boxtimes N_n$ and $\boxplus^m N_n$ are poset matrices. By the definition of ordinal product of poset matrices, the (i, j)-th block Q_{ij} of the matrix $C_m \boxtimes N_n = Q_{m \times n} = [Q_{ij}], 1 \leq i, j \leq m$ takes the following form:

$$Q_{ij} = \begin{cases} N_n \text{ if } i = j, \\ O_n \text{ if } i < j, \\ Z_n \text{ otherwise.} \end{cases} \tag{4}$$

By Theorem 3.4.2, the (i, j)-th block T_{ij} of the matrix $\boxplus_{k=1}^m N_{n_k} = T_t = [T_{ij}]$, $1 \leq i, j \leq m$, where $t = \sum_{i=1}^m n_i$, takes the following form:

$$T_{ij} = \begin{cases} N_{n_i} \text{ if } i = j, \\ O_{n_i, n_j} \text{ if } i < j, \\ Z_{n_j, n_i} \text{ otherwise.} \end{cases} \tag{5}$$

Then for $n_i = n, 1 \leq i \leq m$, Equation 4 and Equation 5 show that $\boxplus^m N_n = T_{m \times n} = Q_{m \times n}$. This implies $C_m \boxtimes N_n = \boxplus^m N_n$.

Now we show that $\mathbf{C}_m \otimes \mathbf{B} \cong \oplus^m \mathbf{B}$. Theorem 3.6.1 shows that $C_m \boxtimes N_n$ represents the poset $\mathbf{C}_m \otimes \mathbf{B}$ and Theorem 3.4.2 shows that $\boxplus^m N_n$ represents the poset $\oplus^m \mathbf{B}$. Then $C_m \boxtimes N_n = \boxplus^m N_n$ implies $\mathbf{C}_m \otimes \mathbf{B} \cong \oplus^m \mathbf{B}$. ∎

3.7. Composition of poset matrices

Definition 3.7.1 The *composition* of the matrix $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and the matrix N_{n_r}, $1 \leq r \leq m$, denoted by $M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$, is a block matrix defined as follows:

$$M_m[N_{n_1},...,N_{n_m}] = \begin{bmatrix} a_{11}N_{n_1} & a_{12}O_{n_1,n_2} & \cdots & a_{1m}O_{n_1,n_m} \\ a_{21}O_{n_2,n_1} & a_{22}N_{n_2} & \cdots & a_{2m}O_{n_2,n_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}O_{n_m,n_1} & a_{m1}O_{n_m,n_2} & \cdots & a_{mm}N_{n_m} \end{bmatrix}$$

Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and N_{n_r}, $1 \leq r \leq m$ be poset matrices. Since M_m is a (0,1)-matrix, the (i, j)-th block Q_{ij} of the block matrix $M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}] = [Q_{ij}]$, $1 \leq i, j \leq m$ can be expressed as follows:

$$Q_{ij} = \begin{cases} N_{n_i} \text{ if } i = j, \\ O_{n_i,n_j} \text{ if } i < j \text{ and } a_{ij} = 1, \\ Z_{n_i,n_j} \text{ if } i < j \text{ and } a_{ij} = 0, \\ O_{n_j,n_i} \text{ if } i > j \text{ and } a_{ij} = 1, \\ Z_{n_j,n_i} \text{ if } i > j \text{ and } a_{ij} = 0. \end{cases} \tag{6}$$

The following example shows the composition of the poset matrices L, C_2, N, and $L^{'}$ (see Example 3.1.1, Example 3.1.3, and Example 3.2.1).

Example 3.7.1

$$L[C_2, N, L^{'}] = \left[\begin{array}{cc|cccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We observe that the block matrix $L[C_2, N, L^{'}]$ is a poset matrix and it represents the poset $\mathbf{B}_{2,1}[\mathbf{C}_2,\mathbf{Z}_4, \mathbf{B}_{1,2}]$ that can be checked immediately from the Hasse diagram shown in the Figure 3.7. We establish this result in the following which gives an association of the composition of poset matrices to posets.

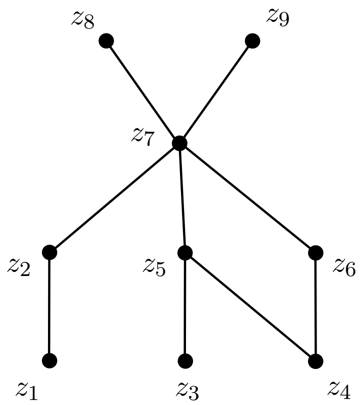


FIGURE 3.7. Hasse diagram of $\mathbf{B}_{2,1}[\mathbf{C}_2,\ \mathbf{Z}_4,\ \mathbf{B}_{1,2}]$ with labeling.

Theorem 3.7.1 Let M_m represents the poset $\mathbf{A}$ and N_{n_i} represents the poset $\mathbf{B}_i$, $1 \leq i \leq m$. Then the matrix $M_m[N_{n_1},\ N_{n_2},\ \ldots,\ N_{n_m}]$ is a poset matrix and it represents the poset $\mathbf{A}[\mathbf{B}_1,\ \mathbf{B}_2,\ \ldots,\ \mathbf{B}_m]$.

Proof. Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$, $N_{n_r} = [b_{ij}]$, $1 \leq i, j \leq n_r$ and $1 \leq r \leq m$. Also let $M_m[N_{n_1},\ N_{n_2},\ \ldots,\ N_{n_m}] = Q_T = [q_{ij}]$, $1 \leq i, j \leq T$, where $T = \sum_{r=1}^{m} n_r$, with block representation $[Q_{ij}]$, $1 \leq i, j \leq m$. Since M_m and N_{n_r}, $1 \leq r \leq m$ are all upper triangular matrices with 1s in the main diagonal, $Q_{ij} = Z_{n_i,n_j}$ for all $i > j$. Thus Q_T is upper triangular with elements 1s in the main diagonal and hence Q_T is clearly reflexive and antisymmetric. For transitivity of Q_T, let $q_{ij} = q_{jk} = 1$ for some $1 \leq i \leq j \leq k \leq T$. Then we have the following cases.

(1) $q_{ij}, q_{jk} \in Q_{rr} = N_{n_r}$ for some $1 \leq r \leq m$. Then there exist $b_{i'j'}, b_{j'k'}, b_{i'k'} \in N_{n_r}$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since N_{n_r} is transitive, $q_{ik} = b_{i'k'} = 1$.

(2) $q_{ij} \in Q_{rs} = O_{n_r,n_s}$ and $q_{jk} \in Q_{ss} = N_{n_s}$ for some $1 \leq r < s \leq m$. Then $q_{ik} \in Q_{rs} = O_{n_r,n_s}$ and clearly $q_{ik} = 1$.

(3) $q_{ij} \in Q_{rs} = O_{n_r,n_s}$ and $q_{jk} \in Q_{st} = O_{n_s,n_t}$ for some $1 \leq r < s < t \leq m$. Then $q_{ik} \in Q_{rt}$. Then by the definition of composition of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since M_m is transitive, $a_{rt} = 1$. Therefore, $Q_{rt} = O_{n_r,n_t}$ and clearly $q_{ik} = 1$.

Thus Q_T is transitive and hence a poset matrix.

Now we show that Q_T represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m]$. Let $A = \{x_1, x_2, ..., x_m\}$ and $B_r = \{y_{t+i} : 1 \leq i \leq n_r\}$ where $t = \sum_{k=1}^{r-1} n_k$. Let $q_{ij} = 1$ in Q_T for some $1 \leq i \leq j \leq T$. Then $q_{ij} \in Q_{rs}$ for some $1 \leq r \leq s \leq m$ and we have the following two cases:

(1) $r = s$. Then $Q_{rs} = N_{n_r}$ and $b_{i^{'}j^{'}} = q_{ij} \in Q_{kl} = N_{n_r}$ for $t = \sum_{k=1}^{r-1} n_k$, $i^{'} = i - t$ and $j^{'} = j - t$. Since $b_{i^{'}j^{'}} = 1$ and N_{n_r} represents $\mathbf{B}_r$, we have $y_{t+i'} \leqslant_{B_r} y_{t+j'}$. Then by the definition of composition of posets, $y_i \leqslant_c y_j$.

(2) $r < s$. Then $Q_{rs} = O_{n_r,n_s}$ for $\sum_{k=1}^{r-1} n_k = t < l = \sum_{k=1}^{s-1} n_k$. Then $y_{t+i'} \in B_r$ and $y_{l+j'} \in B_s$. Then by the definition of composition of poset matrices, $1 = a_{rs} \in M_m$. Sine M_m represents $\mathbf{A}$, we have $x_r \leqslant_A x_s$. Then by the definition of composition of posets, $y_i \leqslant_c y_j$.

For the converse, we show similarly that $y_i \leqslant_c y_j$ implies $1 = q_{ij} \in Q_T$ for all $1 \leq i, j \leq T$. Hence Q_T represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. ∎

The following results show that the composition of poset matrices generalizes the direct sum and the ordinal sum of poset matrices.

Lemma 3.7.2 Let M_{m_i}, $1 \leq i \leq n$ be any poset matrices. Then

(1) $I_n[M_{m_1}, M_{m_2}, ..., M_{m_n}] = \bigoplus_{i=1}^{n} M_{m_i}$

(2) $C_n[M_{m_1}, M_{m_2}, ..., M_{m_n}] = \boxplus_{i=1}^{n} M_{m_i}$

Proof. The proofs are immediate by the definitions of the direct sum, ordinal sum, and composition of poset matrices. ∎

Theorem 3.7.3 Every series-parallel poset is decomposable.

Proof. Let $\mathbf{S}$ be any series-parallel poset. If $|S| < 3$ then trivially $\mathbf{S}$ is decomposable. Otherwise, there exist the posets $\mathbf{A}$ and $\mathbf{B}$, where at least one poset is nonsingleton, such that either $\mathbf{S} \cong \mathbf{A} + \mathbf{B}$ or $\mathbf{S} \cong \mathbf{A} \oplus \mathbf{B}$.
Let M_m represents the poset $\mathbf{A}$ and N_n represents the poset $\mathbf{B}$. Then by Theorem 3.3.1, $M_m \oplus N_n$ is a poset matrix and it represents the poset $\mathbf{A} + \mathbf{B}$, and by Theorem 3.4.1, $M_m \boxplus N_n$ is a poset matrix and it represents the poset $\mathbf{A} \oplus \mathbf{B}$. Also, by Theorem 3.7.1, $I_2[M_m, N_n]$ and $C_2[M_m, N_n]$ are poset matrices and represent

the posets $\mathbf{I}_2[\mathbf{A}, \mathbf{B}]$ and $\mathbf{C}_2[\mathbf{A}, \mathbf{B}]$, respectively. Then, by Lemma 3.7.2, we have the equalities $M_m \oplus N_n = I_2[M_m, N_n]$ and $M_m \boxplus N_n = C_2[M_m, N_n]$ which imply $\mathbf{A}+\mathbf{B} \cong \mathbf{I}_2[\mathbf{A}, \mathbf{B}]$ and $\mathbf{A}\oplus\mathbf{B} \cong \mathbf{C}_2[\mathbf{A}, \mathbf{B}]$, respectively. Hence every series-parallel poset is decomposable. ∎

Proposition 3.6.2 is shown by using the fact that ordinal product of poset matrices gives a generalization of ordinal sum of poset matrices. Below we show that composition of poset matrices generalizes the ordinal product of poset matrices.

Lemma 3.7.4 Let M_m and N_n be poset matrices. Then

$$M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}] = M_m \boxtimes N_n \tag{7}$$

Proof. Substitute $n_i = n$, $1 \leq i \leq m$ in the expression for Q_{ij} in Equation 6. Then (i, j)-th block Q_{ij} of $M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}] = [Q_{ij}]$, $1 \leq i, j \leq m$ takes the following form.

$$Q_{ij} = \begin{cases} N_n \text{ if } i = j, \\ O_{n,n} \text{ if } i < j \text{ and } a_{ij} = 1, \\ Z_{n,n} \text{ if } i < j \text{ and } a_{ij} = 0, \\ O_{n,n} \text{ if } i > j \text{ and } a_{ij} = 1, \\ Z_{n,n} \text{ if } i > j \text{ and } a_{ij} = 0. \end{cases}$$

This implies

$$Q_{ij} = \begin{cases} N_n \text{ if } i = j, \\ O_n \text{ if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n \text{ otherwise.} \end{cases}$$

This equals the expression for Q_{ij} in Equation 3. Thus the (i, j)-th block of the poset matrix $M_m[\underbrace{N_n, N_n, \ldots, N_n}_{m \text{ times}}]$ equals the (i, j)-th block of the poset matrix $M_m \boxtimes N_n$ for all $1 \leq i, j \leq m$. Hence the equality in Equation 7 holds. ∎

Theorem 3.7.5 Every composite poset is decomposable.

Proof. Let $\mathbf{C}$ be any composite poset. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. Let $|A| = m$. Then to show that $\mathbf{C}$ is decomposable we just show the following.

$$\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}[\underbrace{\mathbf{B}, \mathbf{B}, \ldots, \mathbf{B}}_{m \text{ times}}] \tag{8}$$

Let M_m represents the poset $\mathbf{A}$ and N_n represents the poset $\mathbf{B}$. Then by Theorem 3.6.1, $M_m \boxtimes N_n$ is a poset matrix and it represents the poset $\mathbf{A} \otimes \mathbf{B}$, and by Theorem 3.7.1, $M_m[\underbrace{N_n, N_n, \ldots, N_n}_{m \text{ times}}]$ is a poset matrix and it represents the poset $\mathbf{A}[\underbrace{\mathbf{B}, \mathbf{B}, \ldots, \mathbf{B}}_{m \text{ times}}]$. Therefore, the isomorphism in Equation 8 holds by the equality in Equation 7 as we established in Lemma 3.7.4. ∎

CHAPTER 4

Matrix Recognitions of Decomposable Posets

A common problem in the theory of mathematical structures is to recognize those classes of structures which satisfy some common structural properties. As a result, different methods for the recognition of various classes of posets and graphs are considered by numerous authors. Specifically, to reduce complexities of the methods for solving many optimization problems on the structure theory, these begin with some decomposition techniques. These decomposition techniques are used to separate a bigger structure into smaller ones. Therefore, among the frequently studied classes of computationally tractable posets, the classes of decomposable posets play an important role. In this chapter, we consider some classes of decomposable posets and give their matrix recognitions by using the poset matrix.

Fulkerson and Gross [**19**] characterized interval graphs as graphs whose dominant clique-vertex incidence matrix has the property of perfect 1s for columns. Roberts [**48, 49**] characterized proper interval graphs as graphs whose augmented adjacency matrix has perfect 1s property for columns. Tucker [**57**] characterized circular-arc graphs and proper circular-arc graphs by using the properties of perfect 0s, circular 1s, and circularly compatible 1s defined on the augmented adjacency matrix. These classical results give us the idea of defining some properties related to the block of 0s and block of 1s on a poset matrix. We define the properties of block of 0s, block of 1s, and complete block of 1s on a poset matrix and give the matrix recognitions of the P-graphs, P-series, and series-parallel posets. We also define the properties of transitive blocks of 1s and transitive blocks of poset matrices on a block poset matrix and give the matrix recognitions of the factorable posets, composite posets, and decomposable posets.

In Section 4.1, we define the property of complete blocks of 1s on a poset matrix and give a matrix recognition of the P-graphs.

In Section 4.2, we define the property of block of 0s on a poset matrix and give a matrix recognition of the P-series.

In Section 4.3, we define the property of block of 1s on a poset matrix and recall the definition of the property of block of 0s, and give a matrix recognition of the series-parallel posets.

In Section 4.4, we define the property of transitive blocks of poset matrices on a block poset matrix and give a matrix recognition of the factorable posets.

In Section 4.5, we define the property of transitive blocks of 1s on a block poset matrix and give a matrix recognition of the composite posets.

In Section 4.6, we give a generalization to the property of transitive blocks of 1s on a block poset matrix and give, in general, a matrix recognition of the decomposable posets.

4.1. Recognition of P-graphs

Recall that by a poset matrix we mean a poset matrix in upper triangular form. We begin with defining the property of complete blocks of 1s on a poset matrix.

Definition 4.1.1 Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ be a poset matrix. Then we say that M_m has the property of *complete blocks of 1s* of length $\{r_1, r_2, \ldots, r_n\}$, where $0 \leq r_1 < r_2 < \cdots < r_n < m$, if and only if for all $1 \leq i < j \leq m$,

$$a_{ij} = \begin{cases} 1 \text{ if } 1 \leq i \leq r_k \text{ and } r_k + 1 \leq j \leq m \ (1 \leq k \leq n), \\ 0 \text{ otherwise.} \end{cases} \tag{9}$$

Example 4.1.1

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, the poset matrices B, C_4, and D satisfy the property of complete blocks of 1s of lengths $\{1\}$, $\{1, 2, 3\}$, and $\{1, 4\}$, respectively. Also, for every $n \geq 1$, the poset matrix I_n satisfies the property of complete blocks of 1s of length $\{0\}$ and C_n satisfies the property of complete blocks of 1s of length $\{1, 2, \ldots, n-1\}$. We observe that $B = 1 \boxplus I_2$, $C_4 = 1 \boxplus 1 \boxplus 1 \boxplus 1$, and $D = 1 \boxplus I_3 \boxplus 1$. Thus the poset matrices that satisfy the property of complete blocks of 1s of some lengths can be expressed as the ordinal sum of the identity matrices. In the following, we prove this result in general.

Theorem 4.1.1 A poset matrix $M_m \neq I_m$ satisfies the property of complete blocks of 1s if and only if $M_m = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$ for some m_i, $1 \leq i \leq n$.

Proof. Let $M_m = [a_{uv}]$, $1 \leq u, v \leq m$ satisfies the property of complete blocks of 1s of length $\{r_1, r_2, \ldots, r_{n-2}, r_{n-1}\}$. Let $m_0 = 0$, $m_1 = r_1$, $m_{i+1} = r_{i+1} - r_i$ $(1 \leq i \leq n-2)$ and $m_n = m - r_{n-1}$. Since $a_{uv} = 1$ for all $1 \leq u \leq r_i, r_i + 1 \leq v \leq m$, the block $[a_{uv}], m_{i-1} + 1 \leq u \leq m_i, r_i + 1 \leq v \leq m$ can be considered, for every $1 \leq i \leq n-1$, as an m_i-by-$(m - r_i)$ matrix of entries 1s only, that is, $[a_{uv}] = O_{m_i, m-r_i}$. Then for every $1 \leq i \leq n-1$, we have $O_{m_i, m_{i+j}}$, $1 \leq j \leq n-i$ as the horizontal partitions of the augmented matrix $O_{m_i, m-r_i}$. Therefore, by the construction shown in Theorem 3.4.2, we have $M_m = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$. Then by Equation 9, we have $M_{m_i} = I_{m_i}$, $1 \leq i \leq n$.

Conversely, let $M_m = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$, $1 \leq i \leq n$. There exist matrices $O_{m_i, m_{i+j}}$, $1 \leq i \leq n-1$, $1 \leq j \leq n-i$ giving the construction of the direct sum shown in Theorem 3.3.2. Then for every $1 \leq i \leq n-1$, we have the augmented matrices O_{m_i, c_i}, where $c_i = \sum_{j=1}^{n-i} m_{i+j}$, having $O_{m_i,m_{i+j}}$, $1 \leq j \leq n-i$ as the horizontal partitions. Then for every $1 \leq i \leq n-1$, the blocks $[a_{uv}]$, $1 \leq u \leq r_i$, $r_i + 1 \leq v \leq m$, where $r_i = \sum_{j=1}^{i} m_j$, can be considered as the block of 1s. This shows that M_m satisfies the property of block of 1s of lengths r_1, r_2, …, r_{n-1}. Since $M_{m_i} = I_{m_i}$, $1 \leq i \leq n$, clearly, the matrix M_m satisfies the property of complete blocks of 1s of length $\{r_1$, r_2, …, r_{n-2}, $r_{n-1}\}$. ∎

We observe that the poset matrices B, C_4, and D, as in Example 4.1.1, represent the complete bipartite poset $\mathbf{B}_{1,2}$, the chain $\mathbf{C}_4$, and the diamond poset $\mathbf{D}_5$ (Figure 2.5), respectively, which are all P-graphs. We also see that the poset matrices G and G', as in the following example, satisfy the complete blocks of 1s of lengths $\{2,3,4\}$ and $\{1,3,5\}$, respectively, and represent the P-graphs $\mathbf{B}_{2,1} \oplus \mathbf{B}_{1,2}$ and $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ (Figure 3.4), respectively. Below we establish these results in general that gives a matrix recognition of the P-graphs.

Example 4.1.2

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 4.1.2 Let the poset matrix M_m represents the poset $\mathbf{P}$. Then $\mathbf{P}$ is a P-graph if and only if M_m satisfies the property of complete blocks of 1s.

Proof. Let $\mathbf{P}$ be a P-graph. Then there exists $n \leq m$ such that $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \cdots \oplus \mathbf{P}_n$, where for every $1 \leq i \leq n$, $\mathbf{P}_i$ is either the singleton poset or an antichain poset. Let for every $1 \leq i \leq n$, M_{m_i} represents the poset $\mathbf{P}_i$, where $m_i = |P_i|$. Then we have $M_m = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$ (Theorem 3.4.2). Since $\mathbf{P}_i$, for every $1 \leq i \leq n$, is either the singleton poset or an antichain poset, $M_{m_i} = I_{m_i}$ and hence M_m satisfies the property of complete blocks of 1s (Theorem 4.1.1).

For the converse, let M_m satisfies the property of complete blocks of 1s of length $\{0\}$, that is, $M_m = I_m$. Then M_m is trivially a P-graph. Otherwise, let $M_m \neq I_m$ satisfies the property of complete blocks of 1s. Then for some m_i, $1 \leq i \leq n$, we have $M_m = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$ (Theorem 4.1.1). Let M_{m_i} represents the poset $\mathbf{P}_i$ for every $1 \leq i \leq n$. Then $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \cdots \oplus \mathbf{P}_n$ (Theorem 3.4.2). Since $M_{m_i} = I_{m_i}$, for every $1 \leq i \leq n$, the poset $\mathbf{P}_i$ is either the singleton poset or an antichain poset. Hence $\mathbf{P}$ is a P-graph. ∎

4.2. Recognition of P-series

We define the property of block of 0s on a poset matrix as follows:

Definition 4.2.1 Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ be a poset matrix. We say that M_m has the property of *block of 0s* of length r, $1 \leq r < m$ if and only if $a_{ij} = 0$ for all $1 \leq i \leq r$ and $r + 1 \leq j \leq m$.

Example 4.2.1

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, the poset matrices H, I_3, and J satisfy the property of block of 0s of length 2, lengths $1, 2$, and lengths $2, 3$, respectively. Also, for every $n \geq 1$, the poset matrix I_n satisfies the property of blocks of 0s of lengths 1, 2, …, $n - 1$. We observe that $H = C_2 \oplus 1$, $I_3 = 1 \oplus 1 \oplus 1$, and $J = C_2 \oplus I_2$. Thus the poset matrices satisfying the property of block of 0s can be expressed as the direct sum of poset matrices. In the following, we prove these results in general.

Theorem 4.2.1 A poset matrix M_m satisfies the property of block of 0s if and only if $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ for some m_i, $1 \leq i \leq n$.

Proof. Let $M_m = [a_{uv}]$, $1 \leq u, v \leq m$ satisfies the property of block of 0s of lengths $r_1, r_2, \ldots, r_{n-1}$. Let $m_0 = 0$, $m_1 = r_1$, $m_{i+1} = r_{i+1} - r_i$ $(1 \leq i \leq n-2)$, and

$m_n = m - r_{n-1}$. Since $a_{uv} = 0$ for all $1 \le u \le r_i$, $r_i + 1 \le v \le m$, the block $[a_{uv}]$, $m_{i-1}+1 \le u \le m_i$, $r_i+1 \le v \le m$ can be considered, for every $1 \le i \le n-1$, as an m_i-by-$(m - r_i)$ matrix of entries 0s only, that is, $[a_{uv}] = Z_{m_i, m-r_i}$. Then, for every $1 \le i \le n-1$, we have $Z_{m_i, m_{i+j}}$, $1 \le j \le n-i$ as the horizontal partitions of the augmented matrix $Z_{m_i, m-r_i}$. Therefore, by the construction shown in the Theorem 3.3.2, we have $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$.

Conversely, let $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$. There exist matrices $Z_{m_i, m_{i+j}}$, $1 \le i \le n-1$, $1 \le j \le n-i$ giving the construction of the direct sum shown in the Theorem 3.3.2. Then for every $1 \le i \le n-1$, we have the augmented matrices Z_{m_i, c_i}, where $c_i = \sum_{j=1}^{n-i} m_{i+j}$, having $Z_{m_i, m_{i+j}}$, $1 \le j \le n-i$ as the horizontal partitions. Then for every $1 \le i \le n-1$, the blocks $[a_{uv}]$, $1 \le u \le r_i$, $r_i + 1 \le v \le m$, where $r_i = \sum_{j=1}^{i} m_j$, can be considered as the block of 0s. This shows that M_m satisfies the property of block of 0s of lengths r_1, r_2, $\ldots$, r_{n-1}. ∎

Remark 4.2.1 A block of 0s of length r in a poset matrix M_m can be viewed as the matrix $Z_{r, m-r}$ embedded on the top-right corner of M_m.

Example 4.2.2

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{(2,3)-relabeling}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = S'$$

$$S' \xrightarrow{\text{(3,4)-relabeling}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = S''$$

We observe that the poset matrices H, I_3, and J (Example 4.2.1) represent the P-series $\mathbf{C}_2 + \mathbf{1}$, $\mathbf{I}_3$, and $\mathbf{C}_2 + \mathbf{I}_2$ (Figure 2.16 and Figure 2.17) where $\mathbf{C}_2 + \mathbf{1}$

and $\mathbf{C}_2 + \mathbf{I}_2$ are nontrivial P-series. Also, we see that the poset matrix S in the above example does not satisfy the property of block of 0s. But the poset matrix S'' obtained by some relabeling of S satisfies the complete block of 0s of length $\{3\}$ and represent the nontrivial P-series $\mathbf{B}_{2,1} + \mathbf{B}_{1,2}$ shown in Figure 3.3. Below we establish this result in general that gives a matrix recognition of the class of all P-series.

Theorem 4.2.2 Let the poset matrix M_m represent the poset $\mathbf{P}$. Then $\mathbf{P}$ is a P-series if and only if M_m can be relabeled in such a form that either M_m satisfies the property of complete blocks of 1s or M_m satisfies the property of block of 0s and every direct term of M_m satisfies the property of complete blocks of 1s.

Proof. Let $\mathbf{P}$ be a P-series. If $\mathbf{P}$ is a P-graph then M_m satisfies the property of complete blocks of 1s (Theorem 4.1.2). If $\mathbf{P}$ is not a P-graph then there exists $n \leq m$ such that $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n$, where for every $1 \leq i \leq n$, the poset $\mathbf{P}_i$ is a P-graph. Let M_{m_i} ($m_i = |P_i|$) represents the poset $\mathbf{P}_i$ for every $1 \leq i \leq n$. Then $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ (Theorem 3.3.2) and hence M_m satisfies the property of block of 0s (Theorem 4.2.1). Moreover, for every $1 \leq i \leq n$, since M_{m_i} represents the P-graph $\mathbf{P}_i$, the poset matrix M_{m_i} satisfies the property of complete blocks of 1s (Theorem 4.1.2).

For the converse, let M_m can be relabeled so that M_m satisfies the property of complete blocks of 1s. Then clearly $\mathbf{P}$ is a P-graph (Theorem 4.1.2) and hence trivially a P-series. Otherwise, let M_m can be relabeled so that M_m satisfies the property of block of 0s and every direct term of M_m satisfies the property of complete blocks of 1s. Then $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ (Theorem 4.2.1), where for every $1 \leq i \leq n$, the poset matrix M_{m_i} satisfies the property of complete blocks of 1s. Clearly, M_{m_i} represents the P-graph $\mathbf{P}_i$ for every $1 \leq i \leq n$ (Theorem 4.1.2) and $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n$ (Theorem 3.3.2). Hence $\mathbf{P}$ can be expressed as the direct sum of P-graphs and hence $\mathbf{P}$ is a P-series. ∎

4.3. Recognition of series-parallel posets

We define the property of block of 1s which is analogous to the property of block of 0s on a poset matrix as follows:

Definition 4.3.1 Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ be a poset matrix. We say that M_m has the property of *block of 1s* of length r, $1 \leq r < m$ if and only if $a_{ij} = 1$ for all $1 \leq i \leq r$ and $r + 1 \leq j \leq m$.

Example 4.3.1

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, the poset matrices K and P satisfy the property of block of 1s of length 1 and lengths 1 and 4, respectively. Also, any poset matrix satisfying the property of complete blocks of 1s satisfies the property of block of 1s. We observe that $K = 1 \boxplus H$ and $P = 1 \boxplus H \boxplus 1$, where the poset matrix H (Example 4.2.1) satisfies the property of block of 0s. Thus the poset matrices that satisfy the property of block of 1s can be expressed as the ordinal sum of poset matrices. In the following, we prove these results in general.

Theorem 4.3.1 A poset matrix M_m satisfies the property of block of 1s if and only if $M_m = M_{m_1} \boxplus M_{m_2} \boxplus \cdots \boxplus M_{m_n}$ for some positive integers m_i, $1 \leq i \leq n$.

Proof. The proof follows from Theorem 4.1.1 and it is analogous to the proof of Theorem 4.2.1. ∎

Remark 4.3.1 A block of 1s of length r in a poset matrix M_m can be viewed as the matrix $O_{r,m-r}$ embedded on the top-right corner of M_m.

Remark 4.3.2 The poset matrix K (Example 4.3.1) satisfies the property of block of 1s of length 1 but it does not satisfy the property of complete blocks of 1s of length $\{1\}$. On other hand, if a poset matrix $M_m \neq I_m$ satisfies the property of complete blocks of 1s of length $\{r_1, r_2, \ldots, r_n\}$ then M_m must satisfy the property of block of 1s of lengths $r_1, r_2, \ldots, r_n$.

We observe that the poset matrices K and P (Example 4.3.1) represent the tree poset $\mathbf{T}_{1,4} \cong \mathbf{1} \oplus (\mathbf{C}_2 + \mathbf{1})$ (Figure 2.10) and the polygonal poset $\mathbf{P}_{2,1} \cong \mathbf{1} \oplus (\mathbf{C}_2 + \mathbf{1}) \oplus \mathbf{1}$ (Figure 2.6), respectively, which are nontrivial series-parallel posets. Further, we observe that $K = 1 \boxplus \{(1 \boxplus 1) \oplus 1\}$ and $P = 1 \boxplus \{(1 \boxplus 1) \oplus 1\} \boxplus 1$. These show that the poset matrices K and P can be expressed as the direct sum and ordinal sum of 1. Below we establish this result in general that gives a matrix recognition of the class of all series-parallel posets.

Theorem 4.3.2 Let the poset matrix M_m represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then $\mathbf{P}$ is series-parallel if and only if M_m can be relabeled in such a form that M_m satisfies either the property of block of 0s or the property of block of 1s and every term (direct or ordinal) until 1 satisfies either the property of block of 0s or the property of block of 1s.

Proof. Let $\mathbf{P} \ncong \mathbf{1}$ be a series-parallel poset. Then there exist at least the posets $\mathbf{P}_1$ and $\mathbf{P}_2$ with $|P_1| + |P_2| = |P|$ such that either $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ or $\mathbf{P} = \mathbf{P_1} \oplus \mathbf{P}_2$. Let $M_{m_{11}}$ and $M_{m_{12}}$, where $m_{1i} = |P_i|, 1 \leq i \leq 2$, represent the posets $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Then either $M_m = M_{m_{11}} \oplus M_{m_{12}}$ or $M_m = M_{m_{11}} \boxplus M_{m_{12}}$ (Theorem 3.3.1 and Theorem 3.4.1). There exists either the matrix $Z_{m_{11},m_{12}}$ (as a block of 0s of length m_{11}) or the matrix $O_{m_{11},m_{12}}$ (as a block of 1s of length m_{11}) as in the constructions of direct sum and ordinal sum (Theorem 4.2.1 and Theorem 4.3.1). This shows that M_m satisfies either the property of block of 0s of length m_{11} or the property of block of 1s of length m_{11}. Since every term (direct or ordinal) of a series-parallel poset is series-parallel, if $P_i \ncong \mathbf{1}, 1 \leq i \leq 2$, we show similarly that for every $1 \leq i \leq 2$, the poset matrix $M_{m_{1i}}$ satisfy either the property of block of 0s of length m_{2i} or the property of block of 1s of length m_{2i}. Continuing the above process we show that every term (direct or ordinal) until 1 satisfies either the block of 0s property or the block of 1s property.

Conversely, let M_m can be relabeled in such a form that it satisfies either the property of block 0s or the property of block of 1s of length $m_1 < m$. Then there exist M_{m_1} and M_{m_2}, where $m_2 = m - m_1$, such that either $M_m = M_{m_1} \oplus M_{m_2}$ or $M_m = M_{m_1} \boxplus M_{m_2}$ (Theorem 4.2.1 and Theorem 4.3.1). Then either $\mathbf{P} = \mathbf{P}_{01} + \mathbf{P}_{02}$ or $\mathbf{P} = \mathbf{P}_{01} \oplus \mathbf{P}_{02}$, where M_{m_1} and M_{m_2} represent the posets $\mathbf{P}_{01}$ and $\mathbf{P}_{02}$ (Theorem 3.3.1 and Theorem 3.4.1), respectively. Since every term (direct or ordinal) $M_{m_i}, 1 \leq i \leq 2$ until 1 satisfies either the block of 0s property or the block of 1s property, we show similarly that there exist the posets $\mathbf{P}_{i1}$ and $\mathbf{P}_{i2}$ $(1 \leq i \leq 2)$ such that $\mathbf{P}_{0i} = \mathbf{P}_{i1} + \mathbf{P}_{i2}$ or $\mathbf{P}_{0i} = \mathbf{P}_{i1} \oplus \mathbf{P}_{i2}$. Continuing the above process, we show that the poset $\mathbf{P}$ can be expressed as the sum of singleton posets using direct sum and ordinal sum. Hence $\mathbf{P}$ is a series-parallel poset. ∎

Remark 4.3.3 The poset matrix N (Example 3.1.1) satisfies neither the property of block of 0s nor the property of block of 1s for any labeling. Also, it represents the zigzag poset $\mathbf{Z}_4$ (Figure 2.17). This shows that $\mathbf{Z}_4$ is not a series-parallel poset.

4.4. Recognition of factorable posets

We define the property of transitive blocks of poset matrices on a block poset matrix as follows:

Definition 4.4.1 Let $M_{m\times n}$ be a poset matrix consisting of the n-by-n blocks M_{ij}, $1 \leq i, j \leq m$ for some $m > 1$ and $n > 1$. Then $M_{m\times n}$ has the property of *transitive blocks of poset matrices* of length $\{m, n\}$ if and only if for all $1 \leq i, j, k \leq m$ the following conditions hold:

(1) $M_{ii} = N_n$, a poset matrix,

(2) $M_{ij} = Z_n$ for $i > j$; and $M_{ij} = N_n$ or $M_{ij} = Z_n$ for $i < j$,

(3) $M_{ij} = M_{jk} = N_n$ implies $M_{ik} = N_n$.

Example 4.4.1

$$W = \begin{bmatrix} 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 1 & 1 & | & 0 & 1 & 1 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \\ - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix} \qquad W' = \begin{bmatrix} 1 & 1 & | & 1 & 1 & | & 1 & 1 \\ 0 & 1 & | & 0 & 1 & | & 0 & 1 \\ - & - & . & - & - & . & - & - \\ 0 & 0 & | & 1 & 1 & | & 1 & 1 \\ 0 & 0 & | & 0 & 1 & | & 0 & 1 \\ - & - & . & - & - & . & - & - \\ 0 & 0 & | & 0 & 0 & | & 1 & 1 \\ 0 & 0 & | & 0 & 0 & | & 0 & 1 \end{bmatrix}$$

Here, both the poset matrices W and W' satisfy the property of the transitive blocks of poset matrices of lengths $\{2, 3\}$ and $\{3, 2\}$, respectively. Also, W represents the ladder poset $\mathbf{L}_3 \cong \mathbf{C}_2 \times \mathbf{C}_3$ (Figure 4.1) with labeling z_i as the i-th element and W' represents the isomorphic ladder poset $\mathbf{L}_3 \cong \mathbf{C}_3 \times \mathbf{C}_2$ (Figure 4.1) with labeling w_i as the i-th element. Note that we can obtain W' through some relabeling of W (where $\bar{W}$ and $\tilde{W}$ are some poset matrices) as follows:

$$W \xrightarrow{\text{(3,4)-relabeling}} \bar{W} \xrightarrow{\text{(2,3)-relabeling}} \tilde{W} \xrightarrow{\text{(4,5)-relabeling}} W'$$

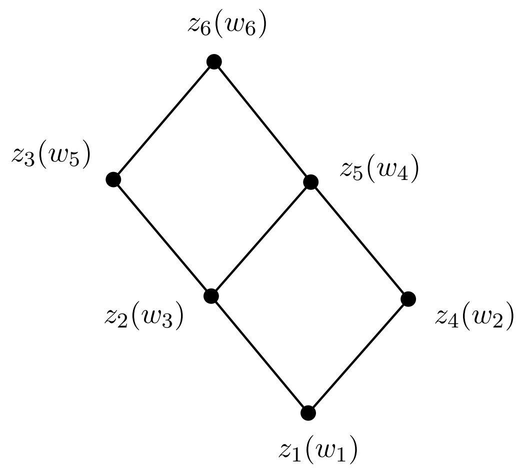


FIGURE 4.1. Hasse diagram of $\mathbf{L}_3$ with labeling.

We observe that $Z = C_2 \otimes C_3$ and $Z' = C_3 \otimes C_2$. We prove this fact in general in the following.

Theorem 4.4.1 A matrix satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$ for some positive integers m and n if and only if it can be obtained as the Kronecker product of some poset matrices M_m and N_n.

Proof. Let the matrix P be obtained as the Kronecker product of the poset matrices M_m and N_n. Then by the definition of Kronecker product, $P = M_m \otimes N_n$, and by Theorem 3.5.1, P is a block poset matrix. This shows that P is upper triangular having the poset matrix N_n as the diagonal blocks satisfying the first two conditions in Definition 4.4.1. Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and $P = [P_{ij}]$, $1 \leq i, j \leq m$ such that $P_{ij} = P_{jk} = N_n$ for some $1 \leq i < j \leq m$. Then by the definition of Kronecker product of poset matrices, we have $a_{ij} = a_{jk} = 1$ (Equation 2). Since M_m is transitive, $a_{ik} = 1$. Therefore, $P_{ik} = N_n$ which satisfies the last condition in Definition 4.4.1. This shows that P satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$.

Conversely, we suppose that the matrix P satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$ for some positive integers m and n and show, similarly, that P can be obtained as the Kronecker product of the poset matrices M_m and N_n. ∎

Example 4.4.2

$$X = \begin{bmatrix} 1 & 1 & 1 & | & 0 & 1 & 0 & | & 0 & 1 & 1 \\ 0 & 1 & 0 & | & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 & | & 0 & 0 & 1 \\ - & - & - & . & - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 1 & | & 0 & 0 & 1 \\ - & - & - & . & - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(5,7)\text{-relabeling}} \begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 1 & 0 & | & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 & | & 0 & 0 & 1 \\ - & - & - & . & - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 & | & 0 & 0 & 1 \\ - & - & - & . & - & - & - & . & - & - & - \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix} = X^{'}$$

We see that although the poset matrix X in the above example seems not to satisfy the property of the transitive blocks of poset matrices, the poset matrix $X^{'}$ obtained by $(5, 7)$-relabeling of X satisfies the property of transitive blocks of poset matrices of length $\{3, 3\}$. Also, $X^{'}$ represents the factorable poset $\mathbf{B}_{2,1} \times \mathbf{B}_{1,2}$ shown in Figure 3.5. We establish this fact in general in the following that gives a matrix recognition of the class of all factorable posets.

Theorem 4.4.2 Let the poset matrix P represent the poset $\mathbf{F}$. Then $\mathbf{F}$ is a factorable poset if and only if P can be relabeled in such a form that it satisfies the property of transitive blocks of poset matrices.

Proof. Let the poset $\mathbf{F}$ be factorable. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{F} \cong \mathbf{A} \times \mathbf{B}$. Let the poset matrix M_m represents $\mathbf{A}$ and N_n represents $\mathbf{B}$. Then by Theorem 3.5.1, the poset matrix $M_m \otimes N_n$ represents the poset $\mathbf{A} \times \mathbf{B} \cong \mathbf{F}$. This shows that the poset matrix P can be relabeled in such a form that $P = M_m \otimes N_n$. Then by Theorem 4.4.1, P satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$.

Conversely, we suppose that the poset matrix P can be relabeled in such a form that it satisfies the property of transitive blocks of poset matrices of length $\{m, n\}$ for some positive integers m and n and show, similarly, that the poset $\mathbf{F}$ is factorable. ∎

4.5. Recognition of composite posets

We define the property of transitive blocks of 1s of length $\{m, n\}$ for some positive integers m and n on a block poset matrix as follows:

Definition 4.5.1 Let $M_{m\times n}$ be a poset matrix consisting of the n-by-n blocks M_{ij}, $1 \le i, j \le m$ for some $m > 1$ and $n > 1$. Then $M_{m\times n}$ has the property of *transitive blocks of 1s* of length $\{m, n\}$ if and only if for all $1 \le i, j, k \le m$ the following conditions hold:

(1) $M_{ii} = N_n$, a poset matrix,

(2) $M_{ij} = Z_n$ for $i > j$; and $M_{ij} = O_n$ or $M_{ij} = Z_n$ for $i < j$,

(3) $M_{ij} = M_{jk} = O_n$ implies $M_{ik} = O_n$.

Example 4.5.1

$$Y = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(3,4)\text{-relabeling}} \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = Y'$$

We observe that although the poset matrix Y, as given in the above example, seems not to satisfy the property of the transitive blocks of 1s, the poset matrix $Y^{'}$, obtained by $(3,4)$-relabeling of Z, satisfies the property of transitive blocks of 1s of the length $\{3,3\}$. This shows that $Y = Y^{'} = L \boxtimes L^{'}$ (Example 3.6.1). We prove this fact in general in the following.

Theorem 4.5.1 A matrix satisfies the property of transitive blocks of 1s of length $\{m,n\}$ for some positive integers m and n if and only if it can be obtained as the ordinal product of some poset matrices M_m and N_n.

Proof. Let the matrix Q be obtained as the ordinal product of the poset matrices M_m and N_n. Then by the definition of the ordinal product of poset matrices, $Q = M_m \boxtimes N_n$, and by Theorem 3.6.1, Q is a block poset matrix. This shows that Q is upper triangular having the poset matrix N_n as the diagonal blocks satisfying the first two conditions in Definition 4.5.1. Let $M_m = [a_{ij}]$, $1 \leq i,j \leq m$ and $Q = [Q_{ij}]$, $1 \leq i,j \leq m$ such that $Q_{ij} = Q_{jk} = O_n$ for some $1 \leq i < j \leq m$. Then by the definition of ordinal product of poset matrices, we have $a_{ij} = a_{jk} = 1$ (Equation 3). Since M_m is transitive, $a_{ik} = 1$. Therefore, $Q_{ik} = O_n$ which satisfies the last condition in Definition 4.5.1. This shows that Q satisfies the property of transitive blocks of 1s of length $\{m,n\}$.

Conversely, we suppose that the matrix Q satisfies the property of transitive blocks of 1s of length $\{m,n\}$ for some positive integers m and n and show, similarly, that the matrix Q can be obtained as the ordinal product of the poset matrices M_m and N_n. ∎

We observe also that the poset matrix $Y^{'}$, as in Example 4.5.1, represents the composite poset $\mathbf{B}_{2,1} \otimes \mathbf{B}_{1,2}$ shown in Figure 3.6. We establish this result in the following where we give a matrix recognition of the class of all composite posets.

Theorem 4.5.2 Let the poset matrix Q represent the poset $\mathbf{C}$. Then $\mathbf{C}$ is a composite poset if and only if Q can be relabeled in such a form that it satisfies the property of transitive blocks of 1s.

Proof. Let the poset $\mathbf{C}$ be a composite poset. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. Let the poset matrix M_m represents $\mathbf{A}$ and the poset matrix N_n represents $\mathbf{B}$. Then by Theorem 3.6.1, the poset matrix $M_m \boxtimes N_n$ represents the poset $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{C}$. This shows that the poset matrix Q can be relabeled in such a form that $Q = M_m \boxtimes N_n$. Then by Theorem 4.5.1, Q satisfies the property of transitive blocks of 1s of length $\{m, n\}$.

Conversely, we suppose that the poset matrix Q can be relabeled in such a form that it satisfies the property of transitive blocks of 1s and show, similarly, that the poset $\mathbf{C}$ is a composite poset. ∎

4.6. Recognition of decomposable posets

In this section we give a generalization to the property of transitive blocks of 1s defined in the previous section. Here, we define the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ for some positive integers m and n_i, $1 \leq m$ on a block poset matrix.

Definition 4.6.1 A poset matrix R is said to have the property of *transitive blocks of 1s* of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ if and only if there exists a block representation $R = [M_{ij}], 1 \leq i, j \leq m$ such that for all $1 \leq i, j, k \leq m$ the following conditions hold:

(1) $M_{ii} = N_{n_i}$, a poset matrix,
(2) $M_{ij} = Z_{n_j,n_i}$ for $i > j$; and $M_{ij} = O_{n_i,n_j}$ or $M_{ij} = Z_{n_i,n_j}$ for $i < j$,
(3) $M_{ij} = O_{n_i,n_j}$ and $M_{jk} = O_{n_j,n_k}$ imply $M_{ik} = O_{n_i,n_k}$.

Note that if $n_1 = n_2 = \cdots = n_m = n$ (say) then we write shortly $\{m, n\}$ for the length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Example 4.6.1

$$Z = \begin{bmatrix} 1 & 0 & | & 1 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 1 & | & 0 & 0 & 1 & 0 & | & 1 & 1 & 1 \\ - & - & . & - & - & - & - & . & - & - & - \\ 0 & 0 & | & 1 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 1 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 0 & 1 & | & 1 & 1 & 1 \\ - & - & . & - & - & - & - & . & - & - & - \\ 0 & 0 & | & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{(2,3)-relabeling}} \begin{bmatrix} 1 & 1 & | & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 1 & | & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ - & - & . & - & - & - & - & . & - & - & - \\ 0 & 0 & | & 1 & 0 & 1 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 1 & 1 & 1 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 1 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 0 & 1 & | & 1 & 1 & 1 \\ - & - & . & - & - & - & - & . & - & - & - \\ 0 & 0 & | & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix} = Z^{'}$$

We observe that although the poset matrix Z in the above example seems not to satisfy the property of the transitive blocks of 1s, the poset matrix $Z^{'}$, obtained from Z by $(2,3)$-relabeling, satisfies the property of transitive blocks of 1s of length $\{3, \{2, 4, 3\}\}$. This shows that $Z = Z^{'} = L[C_2, N, L^{'}]$ (Example 3.7.1). We prove this result in general in the following.

Theorem 4.6.1 A matrix satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ for some positive integers m and n_i, $1 \leq m$ if and only if it can be obtained as the composition of some poset matrices M_m and N_{n_i}, $1 \leq i \leq m$.

Proof. Let the matrix R be obtained as the composition of the poset matrices M_m and N_{n_i}, $1 \leq i \leq m$. Then by the definition of composition, we have $R = M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$, and by Theorem 3.7.1, R is a block poset matrix. This shows that R is upper triangular having the poset matrices N_{n_i}, $1 \leq i \leq m$ as diagonal blocks satisfying the first two conditions in Definition 4.6.1. Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and $R = [R_{ij}]$, $1 \leq i, j \leq m$ such that $R_{ij} = O_{n_i,n_j}$ and $R_{jk} = O_{n_j,n_k}$ for some $1 \leq i < j \leq m$. Then by the definition of composition of poset matrices, we have $a_{ij} = a_{jk} = 1$ (Equation 6). Since M_m is transitive, $a_{ik} = 1$. Thus $R_{ik} = O_{n_i,n_k}$ which satisfies the last condition in Definition 4.6.1. This shows that R satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Conversely, we suppose that the matrix R satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ for some positive integers m and n_i, $1 \leq m$ and show, similarly, that R can be obtained as the composition of some poset matrices M_m and N_{n_i}, $1 \leq i \leq m$. ∎

We observe also that the poset matrix Z', as in Example 4.6.1, represents the decomposable poset $\mathbf{B}_{2,1}[\mathbf{C}_2, \mathbf{Z}_4, \mathbf{B}_{1,2}]$ shown in Figure 3.7. We establish this result in general in the following where we give a matrix recognition of the class of all decomposable posets.

Theorem 4.6.2 Let the poset matrix R represent the poset $\mathbf{D}$. Then $\mathbf{D}$ is decomposable if and only if R can be relabeled in such a form that it satisfies the property of transitive blocks of 1s.

Proof. Let $\mathbf{D}$ be a decomposable poset. There exist the posets $\mathbf{A}$ and $\mathbf{B}_i$, $1 \leq i \leq m$, where $m \geq 2$ and at least one $\mathbf{B}_i$ is nonsingleton, such that $\mathbf{D} \cong \mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. Let M_m represents the poset $\mathbf{A}$ and N_{n_i} represents the posets $\mathbf{B}_i$ for every $1 \leq i \leq m$. Then by Theorem 3.7.1, the matrix $M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$ represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m] \cong \mathbf{D}$. This shows that the poset matrix R can be relabeled in such a form that $R = M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$. Then by Theorem 4.6.1, R satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Conversely, we suppose that matrix R can be relabeled in such a form that it satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ for some positive integers m and n_i, $1 \leq m$ and show, similarly, that poset $\mathbf{D}$ is a decomposable poset. ∎

CHAPTER 5

Exact Enumerations of Unlabeled Posets

For a certain class of mathematical structures, different nontrivial conjectures regarding the recognition of the class can be made by observing variety of examples. In finite cases, this can be done efficiently through computer program by counting and generating, if possible, all the nonisomorphic structures belonging to the class. This is one of the main reasons for which the enumeration of various classes of finite lattices, posets, graphs, and topologies are considered in the literature. In this chapter, we give exact enumerations of the classes of P-graphs, P-series, and series-parallel posets by using the poset matrix.

The class of series-parallel posets, as an important subclass of the repeatedly studied class of decomposable posets, was considered by numerous authors. This class of posets contains the class of P-series as a subclass and the class of P-series contains the class of P-graphs as a subclass. We know that $G(n)$, the number of n-element unlabeled P-graphs, can be given explicitly by the formula $G(n) = 2^{n-1}$ for all $n \geq 1$. Stanley [**54**] gave generating functions for the enumeration of series-parallel posets. Bayoumi et al. [**1**] computed $SP(n)$, the number of n-element unlabeled series-parallel posets, for $n \leq 12$ by recalling the generating function obtained by Stanley [**54**]. Later on, El-Zahar et al. [**15**] gave height counting of $SP(n)$ for $n \leq 15$ by modifying the Stanley's generating function with height of posets as an additional parameter. Unfortunately, neither any explicit formula for $S(n)$, the number of n-element unlabeled P-series, was obtained nor any other methods for determining $S(n)$ were considered. In this chapter, firstly, we give an exact enumeration for unlabeled disconnected posets belonging to a class that is closed under the direct sum of posets. Since the aforementioned classes of decomposable posets are closed under the direct sum, we apply the enumeration method for the unlabeled disconnected posets, and finally, give the enumerations of the classes of P-series and series-parallel posets.

In Section 5.1, we give the matrix recognitions of connected and disconnected P-series and series-parallel posets.

In Section 5.2, we give an exact enumeration of the unlabeled disconnected posets belonging to a class that is closed under the direct sum of posets.

In Section 5.3, we recall the results regarding the matrix recognition of connected and disconnected P-series and give an exact enumeration of the unlabeled P-series.

In Section 5.4, we recall the results regarding the matrix recognition of connected and disconnected series-parallel posets and give an exact enumeration of the unlabeled series-parallel posets.

In Section 5.5, we give the enumeration algorithms and prove their time complexities.

In Section 5.6, we give the numerical results obtained upon implementations of the enumeration algorithms into the computer.

5.1. Recognitions of connected and disconnected posets

In Chapter 4, we defined the properties of block of 0s and block of 1s in a poset matrix and gave matrix recognitions of P-series and series-parallel posets. Here, we recall these results and give, firstly, the recognitions of the connected and disconnected posets by using the poset matrix. Then we give the matrix recognitions of the connected and disconnected P-series and series-parallel posets.

We see that a poset matrix M_3 can satisfy the property of block of 1s of length 1, length 2, and lengths 1, 2. In Example 5.1.1 below, the poset matrix $1 \boxplus I_2$ satisfies the property of block of 1s of length 1, the matrix $I_2 \boxplus 1$ satisfies the property of block of 1s of length 2, and the matrix C_3 satisfies the property of block of 1s of lengths 1, 2.

Example 5.1.1

$$1 \boxplus I_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_2 \boxplus 1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad C_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, M_3 can satisfy the property of block of 0s of length 1, length 2, and lengths 1, 2. In Example 5.1.2 below, the poset matrix $1 \oplus C_2$ satisfies the property of block of 0s of length 1, the matrix $C_2 \oplus 1$ satisfies the property of block of 0s of length 2, and the matrix I_3 satisfies the property of block of 0s of lengths 1, 2.

Example 5.1.2

$$1 \oplus C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad C_2 \oplus 1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that for any labeling, a poset matrix can satisfy one of the two properties at a time but not both the properties together. Now we have the following immediate results regarding the recognitions of the connected posets and disconnected posets.

Theorem 5.1.1 Let M_n represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then $\mathbf{P}$ is disconnected if and only if M_n can be relabeled in such a form that it satisfies the property of block of 0s.

Proof. Proof follows clearly by Theorem 4.2.1 and the definition of disconnected posets. ∎

Note that the above result holds for any subclass of posets that is closed under the direct sum of posets.

Theorem 5.1.2 Let M_n represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then $\mathbf{P}$ is connected if M_n can be relabeled in such a form that it satisfies the property of block of 1s.

Proof. Proof follows clearly by Theorem 4.3.1 and the definition of connected posets. ∎

Note that, in general, the converse of the above result is not true. Because, the zigzag poset $\mathbf{Z_4}$ is connected but the poset matrix N, as in the Example 3.1.1, represents the poset $\mathbf{Z_4}$ and it does not satisfy the property of block of 1s for any labeling. However, the following result shows that the converse of the above result holds in the case of series-parallel posets.

Theorem 5.1.3 Let M_n represent the series-parallel poset $\mathbf{P} \ncong \mathbf{1}$. Then $\mathbf{P}$ is connected if and only if M_n can be relabeled in such a form that it satisfies the property of block of 1s.

Proof. The necessary part of the result follows by Theorem 5.1.2. For the converse, let the poset $\mathbf{P}$ is connected. Since $\mathbf{P}$ is series-parallel, by Theorem 4.3.2, M_n can be relabeled in such a form that it satisfies either the property of block of 0s or the property of block of 1s. If M_n satisfies the property of block of 0s then, by Theorem 5.1.1, it contradicts that $\mathbf{P}$ is connected. Thus, M_n must satisfy the property of block of 1s and hence the condition is sufficient. ∎

Finally, we have the following results regarding the recognitions of connected P-series (equivalently, nontrivial P-graphs), disconnected P-series, connected series-parallel posets, and disconnected series-parallel posets.

Theorem 5.1.4 Let M_n represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then

(1) $\mathbf{P}$ is a connected P-series (equivalently, a nontrivial P-graph) if and only if M_n can be relabeled in such a form that it satisfies the property of complete blocks of 1s of some nonzero lengths.

(2) $\mathbf{P}$ is a disconnected P-series if and only if M_n can be relabeled in such a form that it satisfies the property of block of 0s and every direct term satisfies the property of complete blocks of 1s of some nonzero lengths.

Proof. The proof of the first part follows by Theorem 4.2.2 and Theorem 5.1.1, and the proof of the other part follows by Theorem 4.2.2 and Theorem 5.1.2. ∎

Theorem 5.1.5 Let M_n represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then

(1) $\mathbf{P}$ is connected series-parallel if and only if M_n can be relabeled in such a form that it satisfies the property of block of 1s and every ordinal term until 1 satisfies either the property of block of 0s or the property of block of 1s.

(2) $\mathbf{P}$ is disconnected series-parallel if and only if M_n can be relabeled in such a form that it satisfies the property of block of 0s and every direct term until 1 satisfies either the property of block of 0s or the property of block of 1s.

Proof. The proof of the first part follows by Theorem 4.3.2 and Theorem 5.1.1, and the proof of the other part follows by Theorem 4.3.2 and Theorem 5.1.2. ∎

5.2. Enumeration of unlabeled disconnected posets

In this section, we give an exact enumeration of the unlabeled disconnected posets. For some common methods for the enumeration of posets, we refer the readers to [**7, 10, 25, 29, 31**]. In the most of these cases, the enumeration of a class of posets is done through an algorithmic method where computer programs are used for generating and counting all the pairwise nonisomorphic structures belonging to the class. The running times of these algorithms increase rapidly even though the structures under consideration are enough small in size. Mainly, the running time for generating pairwise nonisomorphic posets makes these algorithms highly time-complex. We observe that in the cases of the algorithms for enumeration of a class of posets closed under the direct sum, the steps for generating pairwise nonisomorphic disconnected posets can be skipped. Therefore, our exact enumeration method for the disconnected posets must help speeding up the aforesaid enumeration process. Here, we firstly establish a criterion for the poset matrices M_n, where $n \geq 2$ is fixed, that satisfy the property of block of 0s and represent a collection of pairwise nonisomorphic posets. Then we give the formulae that give the enumerations of several subcollections of unlabeled disconnected posets following the criterion we established for nonisomorphic direct sum of poset matrices.

5.2.1. Nonisomorphic direct sum criterion.

Since the direct sum of posets is commutative, some M_n that satisfy the property of block of 0s of different lengths can represent isomorphic posets even though every direct term of M_n represents a collection of pairwise nonisomorphic posets. For example, let M_5 satisfy the property of block of 0s of length 2. Then $M_5 = M_2 \oplus M_3$, where M_2 represents the posets $\mathbf{C}_2$ and $\mathbf{I}_2$, and M_3 represents the posets $\mathbf{I}_3$ and $\mathbf{C}_2 + \mathbf{1}$ (Table 5.1). Since the direct sum of posets is commutative, we have

$$\mathbf{I}_2 + (\mathbf{C}_2 + \mathbf{1}) \cong \mathbf{C}_2 + (\mathbf{I}_2 + \mathbf{1}) \cong \mathbf{C}_2 + \mathbf{I}_3.$$

This shows that the poset matrices M_5 that satisfy the property of block of 0s of

TABLE 5.1. Two M_5 that satisfy the property of block of 0s of length 2 and represent isomorphic posets although the corresponding direct terms represent pairwise nonisomorphic posets.

$$C_2 \oplus I_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_2 \oplus (C_2 \oplus 1)$$

length 2 can represent isomorphic posets. Therefore, to establish the criteria for the matrices M_n that satisfy the property of block of 0s and represent pairwise nonisomorphic disconnected posets, we must consider some special cases of the lengths for the property of block of 0s satisfied by M_n. Let $C(n)$ and $D(n)$ denote the number of n-element unlabeled connected posets and disconnected posets, respectively. Clearly, $C(1) = 1$ and $D(1) = 0$. For every $n \geq 2$, we now observe the lengths of the blocks of zeros satisfied by a poset matrix M_n as follows:

(1) M_2 can satisfy the property of block of 0s of length 1 only. Then both the direct terms are M_1. Therefore, $D(2) = 1$.

(2) M_3 can satisfy the property of block of 0s of length 1, length 2, and lengths 1, 2. Here the M_3 satisfying the property of block of 0s of length 1 and the M_3 satisfying the property of block of 0s of length 2 represent isomorphic posets. Therefore, $D(3) = 2$.

(3) All M_4 satisfying the property of block of 0s are given in Table 5.2. Here
   (a) M_4 satisfying the property of block of 0s of length 1 and M_4 satisfying the property of block of 0s of length 3 represent isomorphic posets.
   (b) M_4 satisfying the property of block of 0s of lengths 1, 2 and M_4 satisfying the property of block of 0s of lengths 1, 3 represent isomorphic posets.

   Therefore, $D(4) = 6$. Hasse diagrams of the disconnected posets represented by these poset matrices are given in Figure 5.1.

Therefore, in the case of enumeration of M_n that represent nonisomorphic disconnected posets, to exclude the matrices M_n that satisfy the property of block of 0s and represent isomorphic posets, we restrict, firstly, the lengths in the property of block of 0s to be *nondecreasing inter-distant* as defined below. Secondly, we restrict the direct terms to represent nonisomorphic *connected* posets. Thirdly, we count the *repetition* of the consecutive direct terms of *same order* and reduce the number of M_n accordingly.

TABLE 5.2. All M_4 that satisfy the property of block of 0s and represent pairwise nonisomorphic posets. Here, L and L' are the matrices as in Example 3.1.3 and Example 3.2.1.

$$1 \oplus L' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \oplus L$$

$$1 \oplus C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = C_2 \oplus C_2$$

$$I_2 \oplus C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

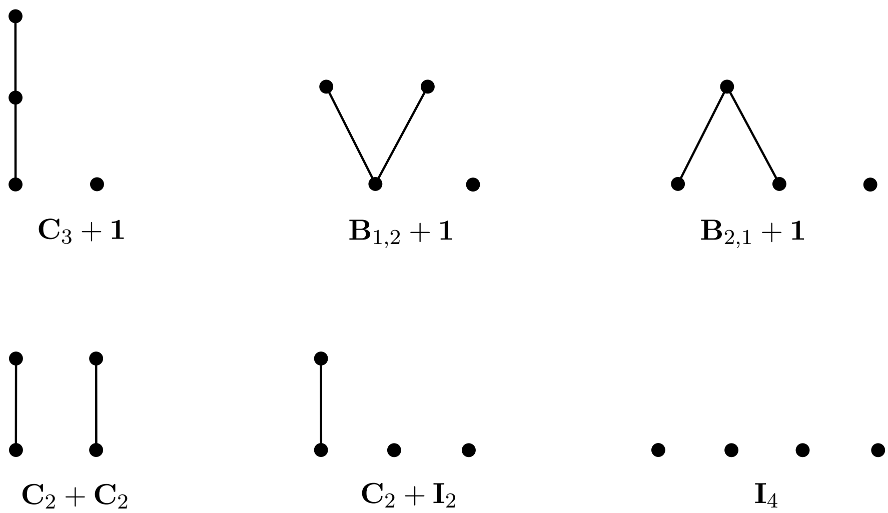


FIGURE 5.1. Hasse diagrams of the disconnected posets represented by the poset matrices given in Table 5.2.

Firstly, for nonisomorphic direct sum, we suppose that M_n satisfies the property of block of 0s of some nondecreasing inter-distant lengths defined as follows:

Definition 5.2.1 The lengths $n_1, n_2, \dots, n_m$, where $1 \le m \le n-1$, chosen as a subcollection of the integers $1, 2, \dots, n-1$ are called

(1) *strictly increasing inter-distant* if $n_1 < n_2 - n_1 < \cdots < n - n_m$,

(2) *equally inter-distant* if $n_1 = n_2 - n_1 = \cdots = n - n_m$,

(3) *nondecreasing inter-distant* if $n_1 \le n_2 - n_1 \le \cdots \le n - n_m$.

For example, all strictly increasing inter-distant, equally inter-distant, and nondecreasing inter-distant lengths $l(m, j)$, where $1 \le m \le 5$ (note that here $n = 6$) and $1 \le j \le p_m$ for some integer p_m, are given in Table 5.3.

TABLE 5.3. All strictly increasing inter-distant (SIID), equally inter-distant (EQID), and nondecreasing inter-distant (NDID) lengths $l(m, j)$, where $1 \le m \le 5$ and $1 \le j \le p$ for some integer p.

		$l(m,j)$		
m	j	SIID	EQID	NDID
1	1	1	3	1
1	2	2	–	2
1	3	–	–	3
2	1	1, 3	2, 4	1, 2
2	2	–	–	1, 3
2	3	–	–	2, 4
3	1	–	–	1, 2, 3
3	2	–	–	1, 2, 4
4	1	–	–	1, 2, 3, 4
5	1	–	1, 2, 3, 4, 5	1, 2, 3, 4, 5

For given $n \ge 2$, to enumerate M_n satisfying the property of block of 0s of some nondecreasing inter-distant lengths and representing pairwise nonisomorphic posets, we comprise the following:

(1) Enumeration of M_n that satisfy the property of block of 0s of some strictly increasing inter-distant lengths and represent nonisomorphic posets.

(2) Enumeration of M_n that satisfy the property of block of 0s of some equally inter-distant lengths and represent nonisomorphic posets.

Secondly, we see that M_n satisfying the property of block of 0s of some nondecreasing inter-distant lengths can represent isomorphic posets if some direct terms represent nonisomorphic disconnected posets. For example, two matrices M_5 that satisfy the property of block of 0s of length 2 and represent isomorphic posets even though the corresponding direct terms M_2 and M_3 of the matrices M_5 represent pairwise nonisomorphic posets are given in Table 5.1.

Considering the above two observations, we now establish the criteria regarding the enumeration of M_n that satisfy the property of block of 0s of nondecreasing inter-distant lengths and represent pairwise nonisomorphic disconnected posets in the following.

Theorem 5.2.1 For $n \geq 2$, let the matrices M_n and M'_n satisfy the property of block of 0s of different nondecreasing inter-distant lengths such that every direct term of M_n and M'_n represents nonisomorphic connected posets only. Then every pair of posets, one represented by M_n and another represented by M'_n, are nonisomorphic.

Proof. For $n \geq 2$ and $1 \leq m, m' \leq n-1$, let M_n and M'_n satisfy the property of block of 0s of the nondecreasing inter-distant lengths $L = \{n_1, n_2, \ldots, n_m\}$ and $L' = \{n'_1, n'_2, \ldots, n'_{m'}\}$, respectively, such that $L \neq L'$. Then we have two different cases as follows:

(1) $m \neq m'$.

In this case, the posets represented by M_n and M'_n have different numbers of direct terms. Then, clearly, every pair of posets, one represented by M_n and another represented by M'_n, are nonisomorphic.

(2) $m = m'$.

For all $0 \leq i \leq m$, say $r_i = n_i - n_{i+1}$ and $r'_i = n'_i - n'_{i+1}$, where we assign $n_0 = n'_0 = 0$ and $n_{m+1} = n'_{m+1} = n$. In this case, since both L and L' contain nondecreasing inter-distant lengths, there exist $0 \leq s, t \leq m$, such that $r_i \neq r'_i$ for all $s \leq i \leq t$, and $r_i = r'_i$ otherwise (in the simplest case). Then, clearly, $M_{r_i} \neq M_{r'_i}$ for all $s \leq i \leq t$. Also, $r_i < r_s$ and $r'_i < r'_s$ for all $0 \leq i \leq s-1$ (when $s > 0$); and $r_i > r_t$ and $r'_i > r'_t$ for all $t+1 \leq i \leq m$. These show that any pair of posets, one represented by M_n and another represented by M'_n, have some direct terms of unequal orders, and hence these are nonisomorphic.

Therefore, in either case, we have every pair of posets, one represented by M_n and another represented by M'_n, are nonisomorphic. ∎

5.2.2. Enumeration formulae for disconnected posets.

Here, we give the enumeration of M_n that satisfy the property of block of 0s of strictly increasing inter-distant lengths and represent pairwise nonisomorphic disconnected posets as follows:

Theorem 5.2.2 Let M_n represent a poset and satisfy the property of block of 0s of strictly increasing inter-distant lengths $n_1, n_2, \ldots, n_t$ such that every direct term M_{r_i}, $1 \leq i \leq t+1$ represents $C(r_i)$ nonisomorphic connected posets. Then $\tilde{D}(n)$, the number of pairwise nonisomorphic disconnected posets represented by M_n, can be given as follows:

$$\tilde{D}(n) = \prod_{i=1}^{t+1} C(r_i) \tag{10}$$

Proof. Since the matrix M_n satisfies the property of block of 0s of lengths $n_1, n_2, \ldots, n_t$, by Theorem 5.1.1, M_n represents disconnected posets and, by Theorem 4.2.1, we have $M_n = M_{n_1} \oplus M_{n_2-n_1} \oplus \cdots \oplus M_{n-n_t}$ for some $M_{r_i} = M_{n_i-n_{i-1}}$, $1 \leq i \leq t+1$ (we assume $n_0 = 0$ and $n_{t+1} = n$) as the direct terms of M_n. Since the lengths $n_1, n_2, \ldots, n_t$ are strictly increasing inter-distant, we have $n_1 < n_2 - n_1 < \cdots < n - n_t$, that is, $r_1 < r_2 < \cdots < r_{t+1}$. Then every direct term M_{r_i}, $1 \leq i \leq t+1$ represents pairwise nonisomorphic connected posets of distinct cardinalities. This shows that M_n represents the pairwise nonisomorphic posets having direct terms as a subcollection of $t+1$ posets each of which is chosen from one of $t+1$ collections of $C(r_i)$ pairwise nonisomorphic posets. Therefore, $\tilde{D}(n)$

equals the number of combination of $t+1$ items each of which is chosen from one of $t+1$ disjoint sets of $C(r_i)$ distinct items. Then, we have $\tilde{D}(n)$ as follows:

$$\tilde{D}(n) = C(r_1) \times C(r_2) \times \cdots \times C(r_{t+1}) = \prod_{i=1}^{t+1} C(r_i)$$

∎

Thirdly, we see that M_n satisfying the property of block of 0s of equally inter-distant lengths can represent isomorphic posets although every direct term represents pairwise nonisomorphic connected posets. All M_6 satisfying the property of block of 0s of length 3 such that both the direct terms M_3 represent pairwise nonisomorphic connected posets and representing pairwise nonisomorphic disconnected posets (shown in Figure 5.2 by using the Hasse diagrams) are given in Table 5.4.

Therefore, by considering the repetition of consecutive direct terms of same order, we establish the result regarding the enumeration of M_n satisfying the property of block of 0s of equally inter-distant lengths and representing pairwise nonisomorphic disconnected posets as follows:

Theorem 5.2.3 Let M_n represent a poset and satisfy the property of block of 0s of equally inter-distant lengths n_1, n_2, ..., n_t such that every direct term M_r represent $C(r)$ pairwise nonisomorphic connected posets. Then $\bar{D}(n)$, the number of pairwise nonisomorphic disconnected posets represented by M_n, can be given as follows:

$$\bar{D}(n) = \binom{C(r)+t}{t+1} \qquad (11)$$

Proof. Since the matrix M_n satisfies the property of block of 0s lengths n_1, n_2, ..., n_t, by Theorem 5.1.1, M_n represents disconnected posets and, by Theorem 4.2.1, we have $M_n = M_{n_1} \oplus M_{n_2-n_1} \oplus \cdots \oplus M_{n-n_t}$ for some $M_{r_i} = M_{n_i-n_{i-1}}$, $1 \le i \le t+1$ (we assume $n_0 = 0$ and $n_{t+1} = n$) as the direct terms of M_n. Since the lengths n_1, n_2, ..., n_t are equally inter-distant, for all $1 \le i \le t+1$, we have $n_i - n_{i-1} = r_i = r$ (say). This shows that all $t+1$ direct terms M_r represents

TABLE 5.4. All M_6 that satisfy the property of block of 0s of length 3 and represent pairwise nonisomorphic disconnected posets. Here, L and L' are the matrices as in Example 3.1.3 and Example 3.2.1.

$$C_3 \oplus C_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = C_3 \oplus L$$

$$C_3 \oplus L' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = L \oplus L$$

$$L \oplus L' = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = L' \oplus L'$$

nonisomorphic connected posets of same cardinality. Thus M_n represents all the nonisomorphic posets that have direct terms as a subcollection of $t+1$ posets each of which is chosen from one of the same $t+1$ collections of $C(r)$ nonisomorphic posets. Therefore, $\bar{D}(n)$ equals the number of combinations of $t+1$ items chosen from $C(r)+t$ distinct items. This gives $\bar{D}(n)$ as follows:

$$\bar{D}(n) = \binom{C(r) + (t+1) - 1}{t+1} = \binom{C(r)+t}{t+1}$$

∎

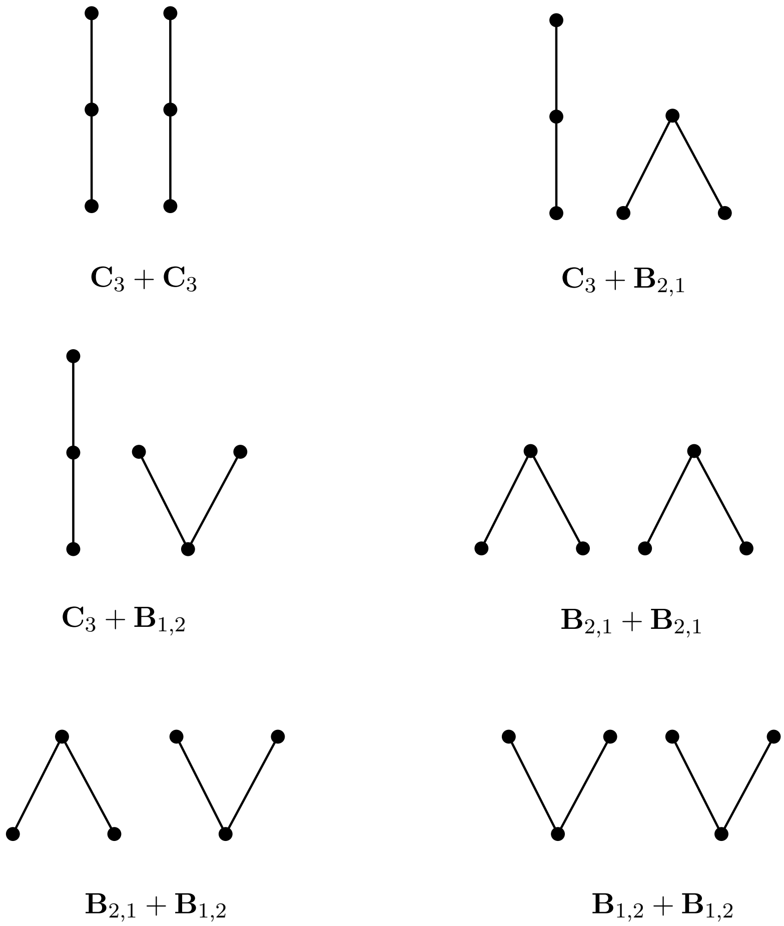


FIGURE 5.2. Hasse diagrams of the disconnected posets represented by the poset matrices given in Table 5.4.

Now we establish the result regarding the enumeration of M_n satisfying the property of block of 0s of nondecreasing inter-distant lengths and representing pairwise nonisomorphic disconnected posets as follows:

Theorem 5.2.4 Let M_n represent a poset and satisfy the property of block of 0s of nondecreasing inter-distant lengths $n_1, n_2, \ldots, n_m$ for some $m \leq n-1$ such that every direct term M_{r_i}, $1 \leq i \leq m+1$ represents $C(r_i)$ nonisomorphic connected posets. Then, there exist r_k, t_k, $1 \leq k \leq q$, where $q \leq m+1$ such that $\tilde{\tilde{D}}(n)$, the number of nonisomorphic disconnected posets represented by M_n, can be given as follows:

$$\tilde{\tilde{D}}(n) = \prod_{k=1}^{q} \binom{C(r_k) + t_k}{t_k + 1} \tag{12}$$

Proof. Let M_n satisfy the property of block of 0s of nondecreasing inter-distant lengths $n_1, n_2, \ldots, n_m$. Then, we have r_k, t_k, $1 \leq k \leq q$, where $q \leq m+1$ as follows:

$$\begin{aligned}
r_1 &= n_1 - n_0 = n_2 - n_1 = \cdots = n_{t_1+1} - n_{t_1}, \\
r_2 &= n_{t_1+2} - n_{t_1+1} = \cdots = n_{t_1+t_2+2} - n_{t_1+t_2+1}, \\
&\vdots \\
r_q &= n_{t_1+\cdots+t_{q-1}+q} - n_{t_1+\cdots+t_{q-1}+q-1} = \cdots = n - n_m.
\end{aligned}$$

Here, $r_1 < r_2 < \cdots < r_q$ and $m = t_1 + \cdots + t_q + q - 1$. We assume $n_0 = 0$ and $n_{m+1} = n$. Then, we have $n_i - n_{i-1} = r_i = r_k$, where $1 \leq k \leq q$ and $t_1 + \cdots + t_{k-1} + k \leq i \leq t_1 + \cdots + t_k + k$. This shows that for every $1 \leq k \leq q$, all t_k+1 consecutive direct terms are M_{r_k} representing $C(r_k)$ pairwise nonisomorphic connected posets of same order. Then for every $1 \leq k \leq q$, by Theorem 5.2.3, we have $\bar{D}((t_k+1)r_k)$, the number of nonisomorphic disconnected posets represented by the poset matrix consisting of $t_k + 1$ consecutive direct terms of order r_k, as follows:

$$\bar{D}((t_k+1)r_k) = \binom{C(r_k) + t_k}{t_k + 1} \tag{13}$$

Therefore, since $r_1 < r_2 < \cdots < r_q$, by Theorem 5.2.2, we have $\bar{\bar{D}}(n)$ as follows:

$$\bar{\bar{D}}(n) = \prod_{k=1}^{q} \bar{D}((t_k+1)r_k) \tag{14}$$

By Equation 13 and Equation 14, we have finally $\bar{\bar{D}}(n)$ as follows:

$$\bar{\bar{D}}(n) = \prod_{k=1}^{q} \binom{C(r_k) + t_k}{t_k + 1}$$

∎

Theorem 5.2.5 Let M_n represent a poset and satisfy the property of block of 0s of lengths $l(m, j)$, where $1 \leq m \leq n-1$ and $1 \leq j \leq p_m$, for some $p_m \leq \binom{n-1}{m}$. Also let $C(r_{mjk})$, the number of $M_{r_{mjk}}$ (the direct terms of M_n) that represent connected posets, and t_{mjk}, the number of the k-th group of consecutive $M_{r_{mjk}}$

of same order r_{mjk}, be given for all $1 \leq k \leq q_{mj}$ for some $q_{mj} \leq m+1$. Then, for $n \geq 2$, we have $D(n)$ as follows:

$$D(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{C(r_{mjk}) + t_{mjk}}{t_{mjk} + 1}, \ n \geq 2 \tag{15}$$

Proof. Let $S(m, j)$ be the number of M_n that satisfy the property of block of 0s of nondecreasing inter-distant lengths $l(m, j)$: $n_{1j}, n_{2j}, \ldots, n_{mj}$, where $1 \leq j \leq p_m \leq \binom{n-1}{m}$, $1 \leq m \leq n-1$. Then, we have r_{mjk}, t_{mjk}, $1 \leq k \leq q_{mj}$, where $q_{mj} \leq m+1$ as follows:

$$\begin{aligned}
r_{mj1} &= n_{ij} - n_{(i-1)j}, 1 \leq i \leq t_{mj1} + 1, \\
r_{mj2} &= n_{ij} - n_{(i-1)j}, t_{mj1} + 2 \leq i \leq t_{mj2} + 1, \\
&\vdots \\
r_{mjq_{mj}} &= n_{ij} - n_{(i-1)j}, t_{mj(q_{mj}-1)} + 2 \leq i \leq t_{mjq_{mj}} + 1.
\end{aligned}$$

Here $r_{mj1} < r_{mj2} < \cdots < r_{mjq_{mj}}$ and we assume $n_{0j} = 0$ and $n_{(m+1)j} = n$. Then the direct terms of M_n are $M_{r_{mjk}}$, $1 \leq i \leq t_{mjk} + 1$, $1 \leq k \leq q_{mj}$. By hypothesis, $M_{r_{mjk}}$ represents $C(r_{mjk})$ nonisomorphic connected posets for every $1 \leq i \leq t_{mjk} + 1$ and $1 \leq k \leq q_{mj}$. Then, by Theorem 5.2.4, we have $S(m, j)$ as follows:

$$S(m, j) = \prod_{k=1}^{q_{mj}} \binom{C(r_{mjk}) + t_{mjk}}{t_{mjk} + 1} \tag{16}$$

Since for every $n \geq 2$, the number $D(n)$ equals the sum of all possible $S(m, j)$ computed for every length $l(m, j)$, where $1 \leq j \leq p_m$ and $1 \leq m \leq n-1$, we have $D(n)$ as follows:

$$D(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} S(m, j), \ n \geq 2 \tag{17}$$

Finally, by Equation 16 and Equation 17, we have $D(n)$ as follows:

$$D(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{C(r_{mjk}) + t_{mjk}}{t_{mjk} + 1}, \ n \geq 2$$

∎

Since all the 6-element disconnected posets are P-series, we see that Example 5.3.1 that illustrates the result in Theorem 5.3.3 also gives an illustration of the result established above in Theorem 5.2.5.

5.3. Enumeration of unlabeled P-series

The class of P-series contains the class of P-graphs as a subclass. Since every nontrivial P-graph (a P-graph except the antichain) is connected, the collection of all the nontrivial P-graphs is actually the collection of all the connected P-series. In this section, firstly, we show by using the poset matrix that the number of all n-element unlabeled connected P-series, can be given by an explicit formula. Then the result regarding the enumeration of n-element unlabeled P-graphs becomes a corollary of this result. Secondly, since the class of all P-series is closed under the direct sum, we recall the result regarding the enumeration of unlabelled disconnected posets established in Section 5.2 and give an exact enumeration of the n-element unlabeled disconnected P-series.

5.3.1. Enumeration of unlabeled connected P-series.

Let $CS(n)$ be the number of unlabeled connected P-series (equivalently, nontrivial P-graphs) with n elements. Clearly, $CS(1) = 1$. For every $n \geq 2$, let the poset matrix M_n represents connected P-series. Then, by Theorem 5.1.4, M_n satisfies the property of complete blocks of 1s of some nonzero lengths. We observe as follows:

(1) An M_2 can satisfy the property of complete blocks of 1s of nonzero length $\{1\}$ only. Then M_2 represents only the connected P-series $\mathbf{C}_2$. Thus $CS(2) = 1$.
(2) An M_3 can satisfy the property of complete blocks of 1s of nonzero lengths $\{1\}$, $\{2\}$, and $\{1, 2\}$. Thus M_3 represents 3 connected P-series all of which are pairwise nonisomorphic. Therefore, we have $CS(3) = 3$.
(3) An M_4 can satisfy the property of complete blocks of 1s of the nonzero lengths given in Table 5.5. This shows that M_4 represents 7 connected P-series all of which are pairwise nonisomorphic. Thus $CS(4) = 7$. All

the connected P-series represented by the these poset matrices are shown in Figure 5.3 by using the Hasse diagrams.

TABLE 5.5. All M_4 satisfying the property of complete blocks of 1s.

$$1 \boxplus 1 \boxplus 1 \boxplus 1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1 \boxplus 1 \boxplus I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \boxplus I_2 \boxplus 1$$

$$I_2 \boxplus 1 \boxplus 1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \boxplus I_3$$

$$I_2 \boxplus I_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_3 \boxplus 1$$

This intuition shows that, for given $n \geq 2$, a poset matrix M_n can satisfy the property of complete blocks of 1s of nonzero lengths equal to a nonempty subset of $\{1, 2, \ldots, n-1\}$ such that in every case it represents a connected P-series. This result gives an explicit formula for enumeration of n-element unlabeled connected P-series (equivalently, nontrivial P-graphs).

Theorem 5.3.1 Let M_n represent a P-series and satisfy the property of block of 1s of all possible lengths. Then for $n \geq 2$, we have $CS(n)$, the number of n-element unlabeled connected P-series, as follows:

$$CS(n) = 2^{n-1} - 1, \; n \geq 2 \tag{18}$$

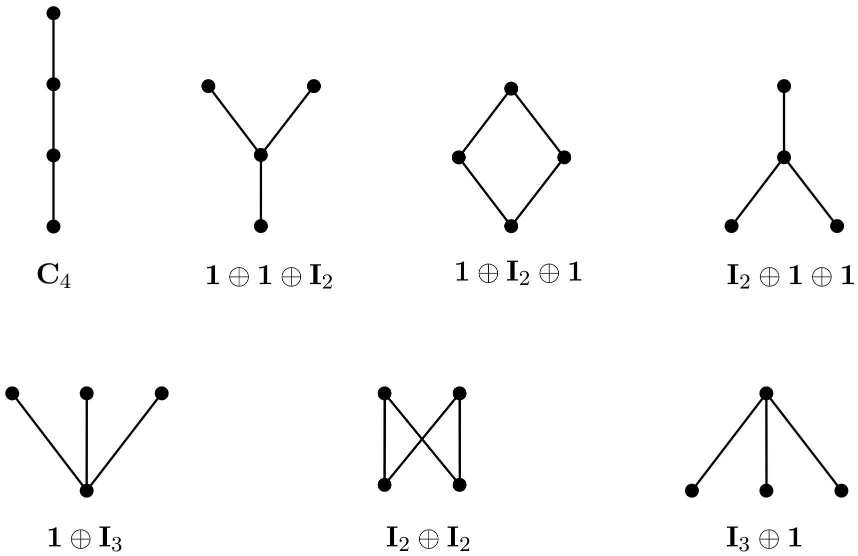


FIGURE 5.3. Hasse diagrams of the connected P-series represented by the poset matrices given in Table 5.5.

Proof. By Theorem 5.1.4, $CS(n)$ equals the number of distinct M_n satisfying the property of complete blocks of 1s of all possible nonzero lengths. We observe that M_n can satisfy the property of complete blocks of 1s of nonzero lengths equal to any nonempty subset of $N = \{1, 2, \ldots, n-1\}$. Since there are $2^{n-1} - 1$ nonempty subsets of N, an M_n can satisfy the property of complete blocks of 1s of lengths equal to one of $2^{n-1} - 1$ nonzero lengths. Thus M_n represents $2^{n-1} - 1$ connected P-series. To show that all the connected P-series represented by M_n are pairwise nonisomorphic, let M_n satisfy the property of complete blocks of 1s of lengths $\{n_1, n_2, \ldots, n_m\}$. Then M_n satisfies the property of block of 1s of lengths $n_1, n_2, \ldots, n_m$ such that the ordinal terms are $I_{n_i - n_{i-1}}$, $1 \le i \le m+1$, where we assume $n_0 = 0$ and $n_{m+1} = n$. Since for every $1 \le i \le m+1$, the ordinal term $I_{n_i - n_{i-1}}$ represents a single poset and the ordinal sum of poset matrices is not commutative, in this case, M_n represents a single connected P-series. Since all the $2^{n-1} - 1$ lengths satisfied by M_n are different, every pair of connected P-series represented by M_n are nonisomorphic. Hence $CS(n) = 2^{n-1} - 1$. ∎

Corollary 5.3.2 For $n \geq 2$, we have $G(n)$, the number of n-element unlabeled P-graphs, as follows:

$$G(n) = 2^{n-1}, \ n \geq 2 \tag{19}$$

Proof. Since for every $n \geq 2$ there are only one trivial P-graph, the n-element antichain $\mathbf{I}_n$, which is excluded by the result in Theorem 5.3.1, we have $G(n) = CS(n) + 1 = 2^{n-1}$. ∎

5.3.2. Enumeration of unlabeled disconnected P-series.

Suppose $DS(n)$, $n \geq 2$ be the number of unlabeled disconnected P-series with n elements. The following result gives the number of unlabeled disconnected P-series according to the number of connected direct terms of the posets.

Theorem 5.3.3 Let M_n represent a P-series and satisfy the property of block of 0s of lengths $l(m, j)$ for all $1 \leq m \leq n-1$ and $1 \leq j \leq p_m$ for some $p_m \leq \binom{n-1}{m}$. Also let $CS(r_{mjk})$, the number of $M_{r_{mjk}}$ (the direct terms of M_n) representing unlabeled connected P-series, and t_{mjk}, the number of the k-th group of consecutive $M_{r_{mjk}}$ of same order r_{mjk}, be given for all $1 \leq k \leq q_{mj}$ for some $q_{mj} \leq m+1$. Then, we have $DS(n)$, $n \geq 2$ as follows:

$$DS(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{2^{r_{mjk}-1} + t_{mjk} - 1}{t_{mjk} + 1}, \ n \geq 2 \tag{20}$$

Proof. Since the class of P-series is closed under the direct sum, by Theorem 5.2.5, we have $DS(n)$ as follows:

$$DS(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{CS(r_{mjk}) + t_{mjk}}{t_{mjk} + 1}, \ n \geq 2 \tag{21}$$

By substituting $CS(n) = 2^{n-1} - 1$, $n \geq 2$ (Equation 18) in the above equation, we have the desired result. Note that for every $n \geq 2$, we have $r_{mjk} \geq 2$ for all $1 \leq m \leq n-1$, $1 \leq j \leq p_m$, and $1 \leq k \leq q_{mj}$. ∎

The following example illustrates the result established in the above theorem.

Example 5.3.1 In this example, we enumerate the 6-element unlabeled disconnected P-series, that is, we determine the number $DS(6)$. We have $CS(r)$, $1 \leq r \leq 5$ (the number of r-element unlabeled connected P-series upto $r = 5$) as follows:

r	1	2	3	4	5
$CS(r)$	1	1	3	7	15

We now compute $S(m, j)$, as in Equation 16, by using the nondecreasing interdistant lengths $l(m, j)$, as in Table 5.3, obtained for all $1 \leq m \leq 5$ and $1 \leq j \leq p_m$, where $p_m \leq \binom{5}{m}$. Recall that we compute the number of unlabeled disconnected posets according to the number of connected direct terms of the posets. Here $m + 1$ equals the number of connected direct terms of a poset.

Number of 6-element unlabeled P-series with 2 connected direct terms:

m	j	$l(m,j)$	$r_{mj1}, \ldots, r_{mj(m+1)}$	$S(m,j)$
1	1	1	$1, 5$	$\binom{1}{1}\binom{15}{1} = 15$
1	2	2	$2, 4$	$\binom{1}{1}\binom{7}{1} = 7$
1	3	3	$3, 3$	$\binom{3+1}{1+1} = 6$
				Total: 28

Number of 6-element unlabeled P-series with 3 connected direct terms:

m	j	$l(m,j)$	$r_{mj1}, \ldots, r_{mj(m+1)}$	$S(m,j)$
2	1	$1, 2$	$1, 1, 4$	$\binom{1}{1}\binom{1}{1}\binom{7}{1} = 7$
2	2	$1, 3$	$1, 2, 3$	$\binom{1}{1}\binom{1}{1}\binom{3}{1} = 3$
2	3	$2, 4$	$2, 2, 2$	$\binom{1+2}{2+1} = 1$
				Total: 11

Number of 6-element unlabeled P-series with 4 connected direct terms:

m	j	$l(m,j)$	$r_{mj1}, \ldots, r_{mj(m+1)}$	$S(m,j)$
3	1	$1, 2, 3$	$1, 1, 1, 3$	$\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{3}{1} = 3$
3	2	$1, 2, 4$	$1, 1, 2, 2$	$\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1} = 1$
				Total: 4

Number of 6-element unlabeled P-series with 5 connected direct terms:

m	j	$l(m,j)$	$r_{mj1},\ldots,r_{mj(m+1)}$	$S(m,j)$
4	1	$1,2,3,4$	$1,1,1,1,2$	$\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1}=1$
				Total: 1

Number of 6-element unlabeled P-series with 6 connected direct terms:

m	j	$l(m,j)$	$r_{mj1},\ldots,r_{mj(m+1)}$	$S(m,j)$
5	1	$1,2,3,4,5$	$1,1,1,1,1,1$	$\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1}=1$
				Total: 1

Thus, $DS(6)=28+11+4+1+1=45$.

5.4. Enumeration of unlabeled series-parallel posets

Since every direct term and ordinal term of a series-parallel poset is also series-parallel, the algorithms for generating series-parallel posets consists of the recursive methods. Therefore, fundamental algorithmic methods for enumeration of this class of posets are ignored by some authors [**15**]. We give an exact enumeration method for the class of series-parallel posets without generating the posets themselves. For this we use the results regarding the recognition of series-parallel posets by using the poset matrix.

To determine the number of n-element unlabeled series-parallel posets, we count all poset matrices M_n that represent pairwise nonisomorphic series-parallel posets. The results established in Theorem 5.1.5 show that the following computations give a complete enumeration of the unlabeled series-parallel posets.

(1) Enumeration of M_n satisfying the property of block of 1s of all possible lengths such that every ordinal term represents pairwise nonisomorphic disconnected series-parallel posets. This calculation gives the number of n-element unlabeled connected series-parallel posets for $n \geq 2$.

(2) Enumeration of M_n satisfying the property of block of 0s of all possible nondecreasing inter-distant lengths such that every direct term represents pairwise nonisomorphic connected series-parallel posets. This calculation gives the number of n-element unlabeled disconnected series-parallel posets for $n \geq 2$.

Let $CSP(n)$ be the number of n-element unlabeled connected series-parallel posets and $DSP(n)$ be the number of n-element unlabeled disconnected series-parallel posets. We have $CSP(1) = 1$ and $DSP(1) = 0$. Although the actual value of $DSP(1) = 0$, for computational purpose, we assume $DSP(1) = 1$. Then for every $n \geq 2$, firstly, we give an exact enumeration formula to determine $CSP(n)$ by using the numbers $DSP(r)$, $1 \leq r \leq n-1$. Secondly, since the class of series-parallel posets is closed under the direct sum, we recall the result regarding the enumeration unlabelled disconnected posets established in Section 5.2 and give an exact enumeration formula to determine $DSP(n)$ by using the numbers $CSP(r)$, $1 \leq r \leq n-1$.

5.4.1. Enumeration of connected series-parallel posets.

Since the ordinal sum of poset matrices is not commutative, the poset matrices M_n satisfying the property of block of 1s represent pairwise nonisomorphic posets if every ordinal term of M_n represents pairwise nonisomorphic posets. We begin with establishing the result regarding the number of M_n satisfying the property of block of 1s for some nonzero lengths and representing pairwise nonisomorphic connected posets.

Lemma 5.4.1 Let for $n \geq 2$, the poset matrix M_n satisfy the property of block of 1s of lengths $n_1, n_2, \ldots, n_m$ for some $1 \leq m \leq n-1$ such that every ordinal term M_{r_i}, $1 \leq i \leq m+1$ represents $P(r_i)$ pairwise nonisomorphic posets. Then $Q(n)$, the number of pairwise nonisomorphic connected posets represented by M_n, can be given as follows:

$$Q(n) = \prod_{i=1}^{m+1} P(r_i) \tag{22}$$

Proof. Since M_n satisfies the property of block of 1s of lengths $n_1, n_2, \ldots, n_m$, by Theorem 5.1.2, M_n represents connected posets. Then, by Theorem 4.3.1, we have $M_n = M_{n_1} \boxplus M_{n_2-n_1} \boxplus \cdots \boxplus M_{n-n_m}$ for some $M_{r_i} = M_{n_i-n_{i-1}}$, $1 \leq i \leq m+1$ (we assume $n_0 = 0$ and $n_{m+1} = n$) as the ordinal terms. Since, for every $1 \leq i \leq m+1$, the ordinal term M_{r_i} represents $P(r_i)$ posets, M_n represents the posets having ordinal terms as a subcollection of $m+1$ posets each of which is chosen from one of $m+1$ collections of $P(r_i)$ posets. Therefore, $Q(n)$ equals the number of collection of $m+1$ items each of which chosen from one of $m+1$ collections of $P(r_i)$ distinct items. Thus

$$Q(n) = P(r_1) \times P(r_2) \times \cdots \times P(r_{m+1}) = \prod_{i=1}^{m+1} P(r_i)$$

Since the ordinal sum of poset matrices is not commutative and the collection of posets represented by every ordinal term of M_n are pairwise nonisomorphic, the above collection of $Q(n)$ posets represented by M_n must be pairwise nonisomorphic. ∎

Theorem 5.4.2 Let M_n represent series-parallel posets and satisfy the property of block of 1s of lengths $l(m,j)$ for all $1 \leq m \leq n-1$ and $1 \leq j \leq \binom{n-1}{m}$. Also let $DSP(r_{mji})$, the number of unlabeled disconnected series-parallel posets represented $M_{r_{mji}}$ (the ordinal terms of M_n) be given for every $1 \leq i \leq m+1$. Then, we have $CSP(n)$, $n \geq 2$ as follows:

$$CSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} \prod_{i=1}^{m+1} DSP(r_{mji}),\ n \geq 2 \tag{23}$$

Proof. For every $1 \leq j \leq \binom{n-1}{m}$ and $1 \leq m \leq n-1$, let $S(m,j)$ be the number of M_n which satisfies the property of block of 1s of lengths $l(m,j)$: $n_{1j}, n_{2j}, \ldots, n_{mj}$, and represents a connected series-parallel posets. Say $r_{mji} = n_{ij} - n_{(i-1)j}$, $1 \leq i \leq m+1$, where we assume $n_{0j} = 0$ and $n_{(m+1)j} = n$. Then the ordinal terms of M_n are $M_{r_{mji}}, 1 \leq i \leq m+1$. By hypothesis, for every $1 \leq i \leq m+1$, the matrix $M_{r_{mji}}$ represents $DSP(r_{mji})$ pairwise nonisomorphic disconnected series-parallel posets. Then, by Lemma 5.4.1, we have $S(m,j)$ for all $1 \leq j \leq \binom{n-1}{m}$ and $1 \leq m \leq n-1$ as follows:

$$S(m,j) = \prod_{i=1}^{m+1} DSP(r_{mji}) \tag{24}$$

Since for every $n \geq 2$, the number $CSP(n)$ equals the sum of all possible $S(m,j)$ computed for every length $l(m,j)$, where $1 \leq j \leq \binom{n-1}{m}$ and $1 \leq m \leq n-1$, we have $CSP(n)$ as follows:

$$CSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} S(m,j), \ n \geq 2 \tag{25}$$

Finally, using Equation 24 in the above equation, we have $CSP(n)$, $n \geq 2$ as follows:

$$CSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} \prod_{i=1}^{m+1} DSP(r_{mji}), \ n \geq 2$$

∎

The following example illustrates the result established in the above theorem.

Example 5.4.1 Enumeration of the 5-element unlabeled connected series-parallel posets, that is, determination of $CSP(5)$. We have $DSP(r)$, $1 \leq r \leq 4$ (the number of r-element unlabeled disconnected series-parallel posets upto $r = 4$) as follows:

r	1	2	3	4
$DSP(r)$	1	1	2	6

For all $1 \leq m \leq 4$ and $1 \leq j \leq p_m$, where $p_m = \binom{4}{m}$, we compute $S(m,j)$ considering the lengths $l(m,j)$, as follows:

Number of 5-element unlabeled connected series-parallel posets with 2 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1}, \ldots, r_{mj(m+1)}$	$S(m,j)$
1	1	1	$1, 4$	$1 \times 6 = 6$
1	2	2	$2, 3$	$1 \times 2 = 2$
1	3	3	$3, 2$	$2 \times 1 = 2$
1	4	4	$4, 1$	$6 \times 1 = 6$
				Total: 16

Number of 5-element unlabeled connected series-parallel posets with 3 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\ldots,r_{mj(m+1)}$	$S(m,j)$
2	1	1, 2	1, 1, 3	$1 \times 1 \times 2 = 2$
2	2	1, 3	1, 2, 2	$1 \times 1 \times 1 = 1$
2	3	1, 4	1, 3, 1	$1 \times 2 \times 1 = 2$
2	4	2, 3	2, 1, 2	$1 \times 1 \times 1 = 1$
2	5	2, 4	2, 2, 1	$1 \times 1 \times 1 = 1$
2	6	3, 4	3, 1, 1	$2 \times 1 \times 1 = 2$
				Total: 9

Number of 5-element unlabeled connected series-parallel posets with 4 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\ldots,r_{mj(m+1)}$	$S(m,j)$
3	1	1, 2, 3	1, 1, 1, 2	1
3	2	1, 2, 4	1, 1, 2, 1	1
3	3	1, 3, 4	1, 2, 1, 1	1
3	4	2, 3, 4	2, 1, 1, 1	1
				Total: 4

Number of 5-element unlabeled connected series-parallel posets with 5 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\ldots,r_{mj(m+1)}$	$S(m,j)$
4	1	1, 2, 3, 4	1, 1, 1, 1, 1	1
				Total: 1

Thus, $CSP(5) = 16 + 9 + 4 + 1 = 30$.

Here, we see that the result proved in Theorem 5.3.1 becomes a corollary of the above result. Because, in the case of $CS(n)$, the number of n-element unlabeled connected P-series, we substitute $I(r)$, the number of r-element antichain posets, for $DSP(r)$ in Equation 23, where we have $I(r_{mji}) = 1$ for all $1 \leq m \leq n-1$, $1 \leq j \leq \binom{n-1}{m}$, and $1 \leq i \leq m+1$. Then we have $CS(n)$ as follows:

$$CS(n) = \sum_{m=1}^{n-1} \binom{n-1}{m} = 2^{n-1} - 1, \; n \geq 2 \tag{26}$$

5.4.2. Enumeration of disconnected series-parallel posets.

The following result gives the number of unlabeled disconnected series-parallel posets according to the number of connected direct terms of the posets.

Theorem 5.4.3 Let M_n represent a series-parallel poset and satisfy the property of block of 0s of lengths $l(m, j)$ for all $1 \leq m \leq n-1$ and $1 \leq j \leq p_m$ for some $p_m \leq \binom{n-1}{m}$. Also let $CSP(r_{mjk})$, the number of $M_{r_{mjk}}$ (the direct terms of M_n) that represent pairwise nonisomorphic connected series-parallel posets, and t_{mjk}, the number of the k-th group of consecutive $M_{r_{mjk}}$ of same order r_{mjk}, be given for all $1 \leq k \leq q_{mj}$ for some $q_{mj} \leq m+1$. Then, for every $n \geq 2$, we have $DSP(n)$ as follows:

$$DSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{CSP(r_{mjk}) + t_{mjk}}{t_{mjk} + 1}, \ n \geq 2 \tag{27}$$

Proof. Since the class of series-parallel posets is closed under the direct sum, the proof follows by Theorem 5.2.5. ∎

5.5. Enumeration Algorithms

Recall that we do not specify the values of the parameters q_{mj}, $1 \leq j \leq p_m$ and p_m, $1 \leq m \leq n-1$, as in Equation 15, explicitly. Therefore, for given n, the computation of $D(n)$, the number of n-element unlabeled disconnected posets, depends mainly on determining the values of these parameters. We have, by inspection, $p_m \leq n^2 \leq \binom{n-1}{m}$ for all $1 \leq m \leq n-1$. Also, $q_{mj} \leq m+1$ for all $1 \leq j \leq p_m$. Here, by Algorithm 5.1, as given below, we determine the aforesaid parameters as well as the number $D(n)$.

The result established in Theorem 5.2.5 gives an exact enumeration of the unlabeled disconnected posets belonging to a class that is closed under the direct sum. Since both the classes of P-series and series-parallel posets are closed under the direct sum, we implement Algorithm 5.1 to compute both $DS(n)$, the number of unlabeled disconnected P-series (Theorem 5.3.3), and $DSP(n)$, the number of unlabeled disconnected series-parallel posets (Theorem 5.4.3), with the number of elements $n \geq 2$.

Moreover, the result in Theorem 5.2.5 establishes that the enumeration of unlabeled posets belonging to a class that is closed under the direct sum depends mainly on the enumeration of unlabeled connected posets belonging to the class. Since, $CS(n)$, the number of n-element unlabeled connected P-series, can be determined by using an explicit formula, firstly, implementation of Algorithm 5.1 only gives a complete determination of the $S(n)$, the number of n-element unlabeled P-series. Secondly, for the enumeration of the class of unlabeled connected series-parallel posets, we apply the result established in Theorem 5.4.2. Here, we use Algorithm 5.2 to determine the parameters in Equation 23 and $CSP(n)$, the number of n-element unlabeled connected series-parallel posets. Finally, by Algorithm 5.3, where Algorithm 5.2 and Algorithm 5.1 are called repeatedly, we compute $SP(n)$, the number of n-elements unlabeled series-parallel posets.

Algorithm 5.1 To compute $D(n)$, where $n \geq 2$ is fixed, the number of n-element unlabeled disconnected posets.

(1) Initialize $D(n)$ as $D(n) = 0$.

(2) Repeat (a) for $m = 1$ to $n - 1$.

- (a) Repeat (i) to (iv) for every distinct nondecreasing inter-distant lengths $l(m, j)$ as is constructed in (i). (Here the total number of repetitions equals the parameter p_m in Equation 15).
  - (i) Construct j-th nondecreasing inter-distant lengths $l(m, j)$ consisting of m integers chosen from the integers less than or equal to $n - 1$.
  - (ii) Initialize $S(m, j)$ as $S(m, j) = 1$ (Equation 16).
  - (iii) Compute t_{mjk} and repeat (A) for every distinct r_{mjk} in the lengths $l(m, j)$. (Here the total number of distinct r_{mjk} equals the parameter q_{mj} in Equation 16).
    - (A) Update $S(m, j)$ with $S(m, j) \times \binom{C(r_{mjk})+t_{mjk}}{t_{mjk}+1}$.
  - (iv) Increase $D(n)$ by $S(m, j)$.

(3) Return $D(n)$.

Lemma 5.5.1 Algorithm 5.1 runs in time $\mathcal{O}(n^5)$.

Proof. The constructions of the lengths $l(m,j)$ in Command (i) have complexity equal to $m(n-1)$. Since $1 \leq t_{mjk}, q \leq m+1$ and $t_{mjk} \propto \frac{1}{q}$, the computations of $S(m,j)$ in Command (iii) have complexity equal to $m+1$. Then $m \leq n-1$ implies that the complexity $m(n-1) \approx \mathcal{O}((n-1)(n-1)) \approx \mathcal{O}(n^2)$ and the complexity $m+1 \approx \mathcal{O}(n-1+1) \approx \mathcal{O}(n)$. Since $1 \leq p \leq n^2$, the repetitions in Command (a) increase the complexity to $n^2\{\mathcal{O}(n^2) + \mathcal{O}(n)\} \approx \mathcal{O}(n^4)$. Finally, the repetitions in Command (2) increase the complexity to $(n-1)\{\mathcal{O}(n^4)\} \approx \mathcal{O}(n^5)$. ∎

Algorithm 5.2 To compute $CSP(n)$, where $n \geq 2$ is fixed, the number of n-element unlabeled connected series-parallel posets.

(1) Initialize $CSP(n)$ as $CSP(n) = 0$.
(2) Repeat (a) for $m = 1$ to $n-1$.
  (a) Repeat (i) to (iv) for every distinct $p = \binom{n-1}{m}$ lengths $l(m,j)$ as is constructed in (i).
    (i) Construct j-th lengths $l(m,j)$ consisting of m integers chosen from the integers less than or equal to $n-1$.
    (ii) Initialize $S(m,j)$ as $S(m,j) = 1$ (Equation 24).
    (iii) Repeat (A) for every $m+1$ distinct r_{mjk} in the lengths $l(m,j)$ (Equation 24).
      (A) Update $S(m,j)$ with $S(m,j) \times D(r_{mjk})$.
    (iv) Increase $CSP(n)$ by $S(m,j)$.
(3) Return $CSP(n)$.

Lemma 5.5.2 Algorithm 5.2 runs in time $\mathcal{O}(n^{m+3})$, where $m \geq 2$ is the number of ordinal terms (either the singleton or disconnected posets) of the posets.

Proof. The constructions of the lengths $l(m,j)$ in Command (i) have complexity equal to $m(n-1)$. Since $1 \leq t_{mjk} \leq m+1$, the computations of $S(m,j)$ in Command (iii) have complexity equal to $m+1$. Then $m \leq n-1$ implies that the complexity $m(n-1) \approx \mathcal{O}(n(n-1)) \approx \mathcal{O}(n^2)$ and the complexity $m+1 \approx \mathcal{O}(n-1+1) \approx \mathcal{O}(n)$. Since $p = \binom{n-1}{m} \leq (\frac{en}{m})^m$, the repetitions in Command (a) increase the complexities to $(\frac{en}{m})^m\{\mathcal{O}(n^2)+\mathcal{O}(n)\} \approx \mathcal{O}(n^m) \times \mathcal{O}(n^2) \approx \mathcal{O}(n^{m+2})$. Finally, the repetitions in Command (2) increase the complexities to $(n-1)\{\mathcal{O}(n^{m+2})\} \approx \mathcal{O}(n^{m+3})$. ∎

Algorithm 5.3 To compute $SP(n)$, $n \geq 2$, the number of n-element unlabeled series-parallel posets.

(1) Initialize the arrays $CSP(n)$ and $DSP(n)$ as $CSP(1) = DSP(1) = 1$.

(2) Repeat (a) to (b) for $m = 2$ to n.

  (a) Compute $CSP(m)$, as in Equation 23, by using the numbers $DSP(r)$, $1 \leq r \leq m-1$.

  (b) Compute $DSP(m)$, as in Equation 27, by using the numbers $CSP(r)$, $1 \leq r \leq m-1$.

(3) Return the sum of $CSP(n)$ and $DSP(n)$.

Lemma 5.5.3 Algorithm 5.3 runs in time $\mathcal{O}(n^{m+4})$, where $m \geq 2$ is the number of ordinal terms (either the singleton or disconnected posets) of the posets.

Proof. The computations in Step (a) and Step (b) have complexities equivalent to $\mathcal{O}(n^{m+3})$ and $\mathcal{O}(n^5)$, respectively. Therefore, the repetitions in Step (2) increase the complexity to $n(\mathcal{O}(n^{m+3})+\mathcal{O}(n^5))$. Since $m \geq 2$, the complexity then becomes equivalent to $\mathcal{O}(n(n^{m+3})) \approx \mathcal{O}(n^{m+4})$. ∎

5.6. Numerical Results

We implement the enumeration algorithms on an Intel CORE i7 (3.6 Ghz) personal computer and determine $S(n)$, the number of n-element P-series, for all $n \leq 75$ and $SP(n)$, the number of n-element unlabeled series-parallel posets, for all $n \leq 33$. Since $CS(n)$, the number of n-element unlabeled connected P-series, equals $2^{n-1}-1$ for all $n \geq 2$, we include only the numbers $DS(n)$, the number of n-element unlabeled disconnected P-series, and $S(n)$ for all $1 \leq n \leq 75$ (Table 5.6, Table 5.7, and Table 5.8). On the other hand, in the case of n-element unlabeled series-parallel posets, since the enumeration depends mainly on determining $CSP(n)$, the number of n-element unlabeled connected series-parallel posets, we include only the numbers $CSP(n)$ and $SP(n)$ for all $1 \leq n \leq 33$ (Table 5.9 and Table 5.10).

In the cases of both P-series and series-parallel posets, we compute the numbers of unlabeled disconnected posets according to the numbers of the connected direct terms (components) of the posets, and in the case of series-parallel posets, we compute the numbers of unlabeled connected posets according to the numbers of the ordinal terms of the posets. In the appendix of this thesis, we include the details of the numerical results regarding $DS(n)$ for $2 \leq n \leq 50$, $CSP(n)$ for $2 \leq n \leq 33$, and $DSP(n)$ for $2 \leq n \leq 33$ in the tables from Table 5.11 to Table 5.16, from Table 5.17 to Table 5.22, and from Table 5.23 to Table 5.26, respectively.

To determine $S(n)$, the machine took about 1 minute for $n \leq 30$ and 3 minutes for $31 \leq n \leq 55$. A modified version of the computer program consisting of the basic operations with numbers greater than the maximum of unsigned 64-bit integers is used to compute $S(n)$ for all $56 \leq n \leq 75$. Then the same machine took about 1 day to determine $S(n)$ for $56 \leq n \leq 65$, and about 1 week to determine $S(n)$ for $66 \leq n \leq 75$. The numbers $S(n), n \leq 7$ are verified by the direct counting of the Hasse diagrams of the P-series. On the other hand, to determine $SP(n)$, the machine took about 1 second for $n \leq 15$, 1 minute for $16 \leq n \leq 20$, 1 hour for $21 \leq n \leq 25$, 1 day for $26 \leq n \leq 30$, and 1 week for $31 \leq n \leq 33$. The numbers $SP(n)$ for $n \leq 25$ match the numbers obtained in [**15**] and the sequence A003430 in [**53**]. Note that, recently, the numbers $SP(n)$, $CSP(n)$ for $n \leq 1000$, and $DSP(n)$ for $n \leq 100$ are included in the sequences A003430, A007453, and A007454, respectively, in OEIS [**53**] and remarked as computed by J. F. Alcover and A. P. Heinz.

Table 5.6. Numbers of unlabeled P-series up to 40 elements.

n	$DS(n)$	$S(n)$	n	$DS(n)$	$S(n)$
1	0	1	21	19,527,774	20,576,349
2	1	2	22	44,225,665	46,322,816
3	2	5	23	99,885,035	104,079,338
4	6	13	24	225,032,910	233,421,517
5	16	31	25	505,797,776	522,574,991
6	45	76	26	1,134,419,571	1,167,974,002
7	115	178	27	2,539,173,978	2,606,282,841
8	296	423	28	5,672,736,196	5,806,953,923
9	733	988	29	12,650,878,942	12,919,314,397
10	1,801	2,312	30	28,165,845,957	28,702,716,868
11	4,338	5,361	31	62,609,097,765	63,682,839,588
12	10,380	12,427	32	138,963,709,623	141,111,193,270
13	24,531	28,626	33	307,997,202,694	312,292,169,989
14	57,622	65,813	34	681,716,264,252	690,306,198,843
15	134,317	150,700	35	1,506,950,601,322	1,524,130,470,505
16	311,465	344,232	36	3,327,039,564,658	3,361,399,303,025
17	718,297	783,832	37	7,336,744,093,779	7,405,463,570,514
18	1,649,579	1,780,650	38	16,160,563,849,577	16,298,002,803,048
19	3,772,448	4,034,591	39	35,557,970,732,785	35,832,848,639,728
20	8,597,284	9,121,571	40	78,156,122,071,028	78,705,877,884,915

Table 5.7. Numbers of unlabeled P-series with elements $41 \leq n \leq 60$.

n	$DS(n)$	$S(n)$
41	171,613,540,653,843	172,713,052,281,618
42	376,459,661,897,527	378,658,685,153,078
43	825,046,583,681,744	829,444,630,192,847
44	1,806,531,250,566,778	1,815,327,343,588,985
45	3,952,141,384,922,689	3,969,733,570,967,104
46	8,638,765,491,016,575	8,673,949,863,105,406
47	18,867,511,982,595,228	18,937,880,726,772,891
48	41,174,866,181,047,968	41,315,603,669,403,295
49	89,787,245,277,280,689	90,068,720,253,991,344
50	195,646,335,428,736,437	196,209,285,382,157,748
51	426,002,974,320,785,329	427,128,874,227,627,952
52	926,928,308,006,848,323	929,180,107,820,533,570
53	2,015,489,707,468,095,175	2,019,993,307,095,465,670
54	4,379,493,357,644,919,915	4,388,500,556,899,660,906
55	9,510,066,579,982,701,722	9,528,080,978,492,183,705
56	20,637,963,678,847,484,327	20,673,992,475,866,448,294
57	44,759,024,190,763,266,720	44,831,081,784,801,194,655
58	97,013,334,312,263,928,700	97,157,449,500,339,784,571
59	210,148,164,788,960,377,845	210,436,395,165,112,089,588
60	454,955,739,608,420,525,822	455,532,200,360,723,949,309

Table 5.8. Numbers of unlabeled P-series with elements $61 \leq n \leq 75$.

n	$DS(n)$	$S(n)$
61	984391488928541916095	985544410433148763070
62	2128762699907670145467	2131068542916883839418
63	4601004718018487598381	4605616404036914986284
64	9939150065266565553301	9948373437303420329108
65	21459612801312397669373	21478059545386107220988
66	46310057239647934930952	46346950727795354034183
67	99888184391717446946509	99961971368012285152972
68	215348913126803452716198	215496487079393129129125
69	464050509871410483198046	464345657776589836023901
70	999507190001735107935203	1000097485812093813586914
71	2151833389142413506647937	2153013980763130917951360
72	4630599615199840030643380	4632960798441274853250227
73	9960367954166712310469680	9965090320649581955683375
74	21415423429824053319007625	21424868162789792609435016
75	46025078414400880740770376	46043967880332359321625159

TABLE 5.9. Number of unlabeled series-parallel posets up to 20 elements.

n	$CSP(n)$	$SP(n)$	n	$CSP(n)$	$SP(n)$
1	1	1	11	83,950	135,292
2	1	2	12	338,878	546,422
3	3	5	13	1,383,576	2,231,462
4	9	15	14	5,702,485	9,199,869
5	30	48	15	23,696,081	38,237,213
6	103	167	16	99,163,323	160,047,496
7	375	602	17	417,553,252	674,034,147
8	1,400	2,256	18	1,767,827,220	2,854,137,769
9	5,380	8,660	19	7,520,966,100	12,144,094,756
10	21,073	33,958	20	32,135,955,585	51,895,919,734

TABLE 5.10. Number of unlabeled series-parallel posets with elements $21 \leq n \leq 33$.

n	$CSP(n)$	$SP(n)$
21	137,849,390,424	222,634,125,803
22	593,407,692,685	958,474,338,539
23	2,562,695,780,058	4,139,623,680,861
24	11,099,806,544,050	17,931,324,678,301
25	48,206,136,562,750	77,880,642,231,286
26	209,876,865,026,303	339,093,495,674,090
27	915,840,095,739,301	1,479,789,701,661,116
28	4,004,923,697,094,450	6,471,397,502,769,942
29	17,547,807,425,910,789	28,356,225,467,215,817
30	77,027,671,121,229,420	124,477,969,755,162,037
31	338,698,369,075,550,442	547,365,728,574,727,797
32	1,491,674,669,942,837,919	2,410,771,901,260,374,293
33	6,579,403,269,510,266,993	10,633,711,793,122,837,110

Conclusions

This thesis explores a few approaches to the recognition and enumeration of some classes of decomposable posets. In the body of the thesis, we introduce the notion of the poset matrix to represent posets. We define the order relation in a square (0,1)-matrix and give an association of this matrix to the posets. Then we do the following.

(1) We show that the matrix transpose of a poset matrix represents the poset dual to the poset represented by that poset matrix.
(2) We describe some interpretations regarding the interchanges of rows and columns simultaneously in a poset matrix (called relabeling) and the matrix transpose of a poset matrix.
(3) We show that every poset matrix can be relabeled into an upper (equivalently, lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling.
(4) We recall some of the common operations in matrices and introduce the notions of ordinal sum, ordinal product, and a composition of matrices. We establish the algebraic interpretations of the direct sum, ordinal sum, Kronecker product, ordinal product, and composition of poset matrices.
(5) We define the property of block of 1s, block of 0s, and complete blocks of 1s on a poset matrix. Then we give the matrix recognitions of the classes of P-graphs, P-series, and series-parallel posets, factorable posets, composite posets, and decomposable posets by using the poset matrix.
(6) We also define the property of transitive blocks of 1s and transitive blocks of poset matrices on a block poset matrix. Then we give the matrix recognitions of the classes of factorable posets, composite posets, and decomposable posets by using the poset matrix.

(7) We give an exact enumeration of the unlabeled disconnected posets belonging to a class of posets that is closed under the direct sum. Then we give the exact enumerations of the P-series and series-parallel posets by using the results obtained regarding the matrix recognitions of these classes of posets.

(8) We give an algorithm to determine the parameters involved in the enumeration formulae obtained for the exact enumeration of unlabeled disconnected posets as well as to determine the number of unlabeled disconnected P-series and series-parallel posets. We show that the enumeration algorithms run in polynomial time with complexity $\mathcal{O}(n^5)$.

(9) We also give an algorithm to determine the number of unlabeled series-parallel posets. We show that the enumeration algorithm runs in polynomial time with complexity $\mathcal{O}(n^{m+4})$, where $m \geq 2$ is the number of ordinal terms (either the singleton or disconnected) of the posets.

(10) We implement these enumeration algorithms into the computer and obtain numerical results for the number of unlabeled P-series and series-parallel posets up to 73 elements and 33 elements, respectively.

We remark that this research work gives foundations for numerous researches regarding the recognitions and enumerations of various classes of mathematical structures including the following.

(1) The class of unlabeled N-free posets
(2) The class of unlabeled interval order posets
(3) The class of unlabeled posets with bounded decomposition parameter
(4) The class of unlabeled height-balanced rooted trees and forests

For some existing results on the recognition and enumeration of the N-free posets, interval order posets, and the posets with bounded decomposition parameter, please refer to the articles by Habib and Möhring [**23**] and Khamis [**30**].

We believe that our present research will be useful for further researches in the areas considered in this thesis.

List of articles based on this thesis

(1) S. U. Mohammad and M. R. Talukder, *Poset matrix and recognition of series-parallel posets*, International Journal of Mathematics and Computer Science, 15 (2020), no. 1, 107-125.

(2) S. U. Mohammad and M. R. Talukder, *Interpretations of Kronecker product and ordinal product of poset matrices*, International Journal of Mathematics and Computer Science, 16 (2021), no. 4, 1665-1681.

(3) S. U. Mohammad, M. R. Talukder, and S. N. Begum, *Recognition of decomposable posets by using the poset matrix*, Italian Journal of Pure and Applied Mathematics, (Accepted).

(4) S. U. Mohammad, M. R. Talukder, and S. N. Begum, *Enumeration of unlabeled P-series by using the poset matrix* (Submitted).

(5) S. U. Mohammad, M. R. Talukder, and S. N. Begum, *Exact enumeration of unlabeled disconnected posets by using the poset matrix* (Submitted).

(6) S. U. Mohammad, M. R. Talukder, and S. N. Begum, *Enumeration of unlabeled series-parallel posets by using the poset matrix* (Submitted).

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Appendix

In this appendix, we give some pseudocodes that we developed to implement the enumeration algorithm into the computer. Here, we use Pseudocode 5.1 and Pseudocode 5.2 to compute the numbers of n-element unlabeled connected P-series and series-parallel posets, respectively. We use Pseudocode 5.3 to compute the numbers of unlabeled disconnected P-series and series-parallel posets. We also use the function $DisconnectedPosets()$ (Pseudocode 5.3) to compute $DS(n)$, the number of n-element unlabeled P-series, and $DSP(n)$, the number of n-element unlabeled series-parallel posets, where we replace the function $ConnectedPosets()$ (Pseudocode 5.5) by the functions $ConnectedPseries()$ (Pseudocode 5.1) and $ConnectedSpposets()$ (Pseudocode 5.2), respectively.

Here, in Pseudocode 5.3, we use the functions $NonisomLengths()$ (Pseudocode 5.4) and $NonisomDirectsum()$ (Pseudocode 5.5) to compute the nondecreasing inter-distant lengths and the nonisomorphic direct sum of unlabeled connected posets given in Section 5.2.1.

We implement the enumeration algorithms on an Intel CORE i7 (3.6 GHz) personal computer and obtain some numerical results. Since $CS(n)$, the number of n-element unlabeled connected P-series, equals $2^{n-1} - 1$ for all $n \geq 2$, we include only the numbers $DS(n)$, computed according to the number of connected direct terms of the posets for all $2 \leq n \leq 50$, in the tables from Table 5.11 to Table 5.16. In the case of n-element unlabeled series-parallel posets, we include $CSP(n)$, $2 \leq n \leq 33$, computed according to the number of ordinal terms (either the singleton or disconnected), in the tables from Table 5.17 to Table 5.22, and

$DSP(n)$, $2 \leq n \leq 33$, computed according to the number of connected direct terms of the posets, in the tables from Table 5.23 to Table 5.26.

Note that, in the cases of disconnected posets, we omit some rows of the tables, because the numbers in these rows become fixed and can be found from the preceding tables.

Pseudocode 5.1 To count the number of n-element connected P-series.

$CS(n) \leftarrow ConnectedPseries(n)$

Input: n, a positive integer.

Output: $CS(n)$, the number of poset matrices M_n except I_n that represent n-element unlabeled connected P-series, that is, the number of n-element unlabeled P-graphs, based on the result established in Theorem 5.3.1.

begin

(1) **if** $n = 1$

(2) $CS(n) \leftarrow 1;$

(3) **else**

(4) $CS(n) \leftarrow 2^{n-1} - 1;$

(5) **end**

stop

Pseudocode 5.2 To count the number of n-element unlabeled connected series-parallel posets.

$$CSP(n) \leftarrow ConnectedSpposets(n)$$

Input: n, a positive integer greater than 1.

Output: $CSP(n)$, the number of the poset matrices M_n that represent n-element unlabeled connected series-parallel posets.

begin

(1) $CSP(n) \leftarrow 0$ (initializing the number of n-element unlabeled connected posets having 2 or more direct terms);

(2) **for** $m = 1$ to $n - 1$

(3) $\quad$ *lengtharray* $\leftarrow$ the collection of all m positive integers chosen from the integers less than n;

(4) $\quad$ $S_m \leftarrow 0$ (initializing the number of n-element unlabeled connected posets having $(m + 1)$ ordinal terms);

(5) $\quad$ **for** $p = 1$ to $\binom{n-1}{m}$

(6) $\quad\quad$ *lengths* $\leftarrow$ p-th lengths in *lengtharray*;

(7) $\quad\quad$ $S_p \leftarrow 1$ (initializing the number of connected posets represented by M_n that satisfies the property of block of 1s of the lengths in *lengths*);

(8) $\quad\quad$ **for** $k = 1$ to $m + 1$

(9) $\quad\quad\quad$ $r_{mpk} \leftarrow$ the difference of the k-th and $(k+1)$-th lengths;

(10) $\quad\quad\quad$ $S_p \leftarrow S_p \times DSP(r_{mpk})$;

(11) $\quad\quad$ **end**

(12) $\quad\quad$ $S_m \leftarrow S_m + S_p$;

(13) $\quad$ **end**

(14) $\quad$ $CSP(n) \leftarrow CSP(n) + S_m$;

(15) **end**

stop

Pseudocode 5.3 To count the number of n-element unlabeled disconnected posets.

$$D(n) \leftarrow DisconnectedPosets(n)$$

Input: n, a positive integer greater than 1.

Output: $D(n)$, the number of the poset matrices M_n that represent n-element unlabeled disconnected posets.

begin

```
D(n) ← 0 (initializing the number of n-element unlabeled disconnected
              posets having 2 or more direct terms);
for m = 1 to n − 1
    lengtharray ← NonisomLengths(n, m);
    S_m ← 0 (initializing the number of n-element unlabeled
              disconnected posets having (m + 1) direct terms);
    for p = 1 to the number of lengths in lengtharray
        lengths ← p-th lengths in lengtharray;
        (S_p, i) ← NonisomDirectsum(lengths, 1);
        while i ≤ m
            (t, i) ← NonisomDirectsum(lengths, i);
            S_p ← S_p × t;
        end
        S_m ← S_m + S_p;
    end
    D(n) ← D(n) + S_m;
end
```

stop

Pseudocode 5.4 To find m-element nondecreasing inter-distant lengths.

$$lengtharray \leftarrow NonisomLengths(number, m)$$

Input: *number*, an integer such tat the collection of integers are less than this number; and m, an integer giving the number of integers in each subcollection.

Output: *lengtharray*, a p-by-m matrix consisting of p subcollections of m integers giving p nondecreasing inter-distant lengths.

```
begin
    i_m ← the greatest integer ≤ (number)(m/(m+1));
    temparray ← all m-element subcollections of the integers 1, 2, ..., i_m;
    count ← 0;
    for i = 1 to the number of lengths in temparray
        lengths ← i-th lengths in temparray;
        lengths(i_m + 1) ← number (adjoin number to lengths as the last
                                   element);
        diff ← lengths(1);
        flag ← 1;
        for j = 2 to m + 1
            if lengths(j) − lengths(j − 1) < diff
                flag ← 0;
                break;
            else if lengths(j) − lengths(j − 1) > diff
                diff ← lengths(j) − lengths(j − 1);
            end
        end
        if flag = 1
            count ← count + 1;
            lengtharray(count) ← lengths;
        end
    end
stop
```

Pseudocode 5.5 To count the number of nonisomorphic direct sums.

$(nsum, newindex) \leftarrow NonisomDirectsum(lengths, index)$

Input: *lengths*, a subcollection of integers; and *index*, a position index of an integer in the length *lengths*.

Output: *newindex*, a position index of an integer in the length *lengths*; and *nsum*, the number of poset matrices satisfying the property of blocks of 0s of lengths equal to the integers form *index* to *newindex* in the length *lengths*.

```
begin
    if index = 1
        diff ← lengths(index);
    else
        diff ← lengths(index) − lengths(index − 1);
    end
    nsum ← ConnectedPosets(diff);
    j ← index + 1;
    repeat ← 0;
    while j ≤ the number of elements in lengths and lengths(j)
            − lengths(j − 1) = diff
        repeat ← repeat + 1;
        j ← j + 1;
    end
    if repeat > 0
        nsum ← (nsum + repeat) choose (nsum − 1);
    end
    newindex ← j;
stop
```

TABLE 5.11. $DS(n)$ for $2 \leq n \leq 18$ according to the number of connected direct terms $d = m + 1$, where $2 \leq d \leq 18$.

$d \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1	1	4	10	28	67	167	388	908	2053	4629	10246	22566	49159	106567	229384	491656
3		1	1	4	11	31	80	213	534	1343	3291	7980	19040	44984	104988	242884	556791
4			1	1	4	11	32	83	226	580	1504	3796	9536	23583	5,900	140496	338304
5				1	1	4	11	32	84	229	593	1550	3957	10062	25244	62948	155431
6					1	1	4	11	32	84	230	596	1563	4003	10223	25770	64637
7						1	1	4	11	32	84	230	597	1566	4016	10269	25931
8							1	1	4	11	32	84	230	597	1567	4019	10282
9								1	1	4	11	32	84	230	597	1567	4020
10									1	1	4	11	32	84	230	597	1567
11										1	1	4	11	32	84	230	597
12											1	1	4	11	32	84	230
13												1	1	4	11	32	84
14													1	1	4	11	32
15														1	1	4	11
16															1	1	4
17																1	1
18																	1
$DS(n)$	1	2	6	16	45	115	296	733	1801	4338	10380	24531	57622	134317	311465	718297	1649579

Table 5.12. $DS(n)$ for $19 \leq n \leq 27$ according to the number of connected direct terms $d = m + 1$, where $2 \leq d \leq 14$.

$d \diagdown n$	19	20	21	22	23	24	25	26	27
2	1048585	2228489	4718602	9961994	20971531	44041227	92274700	192940044	402653197
3	1266751	2861202	6422009	14329484	31805747	70252549	154488108	338336382	738193444
4	807237	1912072	4494918	10497181	24356462	56184170	128879083	294109263	667901109
5	381277	928112	2244879	5394318	12886346	30605995	72302182	169918416	397378534
6	160626	396947	974151	2377592	5769408	13929199	33459298	79999191	190404747
7	65163	162315	402178	990017	2424639	5906454	14321530	34565725	83075361
8	25977	65324	162841	403867	995248	2440550	5953753	14459900	34963876
9	10285	25990	65370	163002	404393	996937	2445781	5969664	14507254
10	4020	10286	25993	65383	163048	404554	997463	2447470	5974895
11	1567	4020	10286	25994	65386	163061	404600	997624	2447996
12	597	1567	4020	10286	25994	65387	163064	404613	997670
13	230	597	1567	4020	10286	25994	65387	163065	404616
14	84	230	597	1567	4020	10286	25994	65387	163065
$DS(n)$	3772448	8597284	19527774	44225665	99885035	225032910	505797776	1134419571	2539173978

TABLE 5.13. $DS(n)$ for $28 \leq n \leq 34$ according to the number of connected direct terms $d = m + 1$, where $2 \leq d \leq 18$.

$d \diagdown n$	28	29	30	31	32	33	34
2	838864909	1744830478	3623886862	7516192783	15569272847	32212254736	66572025872
3	1605012086	3478467868	7516176493	16195589395	34807097956	74625024265	159629552218
4	1509867489	3398600871	7619289505	17016923183	37870246585	83994797573	185706360360
5	924973200	2143477586	4946037062	11366662041	26020964794	59348001466	134881344515
6	451241226	1064970949	2503508837	5862757276	13679302057	31804663743	73695371416
7	198847778	474135412	1126358220	2666381958	6290646019	14793003401	34678077433
8	84205462	202013276	482899046	1150352137	2731408818	6465188021	15257303293
9	35102561	84605296	203150904	486095681	1159235316	2755839396	6531740146
10	14523165	35149915	84744047	203551123	487235394	1162441405	2764762205
11	5976584	14528396	35165826	84791401	203689874	487635691	1163581580
12	2448157	5977110	14530085	35171057	84807312	203737228	487774442
13	997683	2448203	5977271	14530611	35172746	84812543	203753139
14	404617	997686	2448216	5977317	14530772	35173272	84814232
15	163064	404617	997687	2448219	5977330	14530818	35173433
16	65387	163064	404617	997687	2448220	5977333	14530831
17	25994	65387	163064	404617	997687	2448220	5977334
18	10286	25994	65387	163064	404617	997687	2448220
$DS(n)$	5672736196	12650878942	28165845957	62609097765	138963709623	307997202694	681716264252

TABLE 5.14. $DS(n)$ for $35 \leq n \leq 40$ according to the number of connected direct terms $d = m + 1$, where $2 \leq d \leq 21$.

$d \diagdown n$	35	36	37	38	39	40
2	137438953489	283467907089	584115552274	1202590973970	2473901162515	5085241540627
3	340734006527	725849342543	1543324783860	3275628128836	6940666889448	14683061004856
4	409351418116	899770401861	1972412861594	4312761385740	9407206034047	20472269741866
5	305512784640	689767269943	1552503917727	3483993382458	7796348096160	17399073347974
6	170201538669	391842528114	899355963864	2058114658969	4696439658814	10687375059445
7	81047757227	188867721744	438881655169	1017064309232	2350701482265	5419121237462
8	35902678657	84251790220	197186885086	460325889435	1071959991411	2490307177236
9	15436987973	36383784779	85529867179	200557072734	469150541905	1094913052502
10	6556324949	15504111139	36565492083	86017890423	201858086319	472594806691
11	2767970824	6565259359	15528745095	36632808617	86200321470	202348701371
12	1163981877	2769111090	6568468524	15537682523	36657456541	86267697710
13	487821796	1164120628	2769511387	6569608790	15540891793	36666394606
14	203758370	487837707	1164167982	2769650138	6570009087	15542032059
15	84814758	203760059	487842938	1164183893	2769697492	6570147838
16	35173479	84814919	203760585	487844627	1164189124	2769713403
17	14530834	35173492	84814965	203760746	487845153	1164190813
18	5977334	14530835	35173495	84814978	203760792	487845314
19	2448220	5977334	14530835	35173496	84814981	203760805
20	997687	2448220	5977334	14530835	35173496	84814982
21	404617	997687	2448220	5977334	14530835	35173496
$DS(n)$	1506950601322	3327039564658	7336744093779	16160563849577	35557970732785	78156122071028

TABLE 5.15. $DS(n)$ for $41 \leq n \leq 45$ according to the number of connected direct terms $d = m + 1$, where $2 \leq d \leq 23$.

$d \diagdown n$	41	42	43	44	45
2	10445360463892	21440477265940	43980465111061	90159954526229	184717953466390
3	31015389975772	65420940806699	137805456299215	289904563759646	609129439696065
4	44454957781390	96331619708056	208331276709173	449693716677808	968931536739116
5	38728538760019	85990928093694	190473579856294	420938987475544	928208260681351
6	24255843356810	54908954740940	123990449527175	279310303001575	627730601961794
7	12461643067304	28586943323221	65423791190219	149385168779836	340337759138687
8	5771925395845	13347881861258	30800377251935	70921565825199	162968649866921
9	2549629583160	5924315615251	13737078604595	31788839948556	73418617030197
10	1103971401864	2573304750371	5985827821321	13895997702443	32197202864856
11	473904783197	1107445726719	2582461315091	6009816527499	13958490498912
12	202531369392	474396293016	1108758936042	2585946920021	6019011268778
13	86292349183	202598762187	474579032326	1109250730904	2587261213457
14	36669603876	86301287368	202623414395	474646429244	1109433489576
15	15542432356	36670744142	86304496638	202632352580	474671081588
16	6570195192	15542571107	36671144439	86305636904	202635561850
17	2769718634	6570211103	15542618461	36671283190	86306037201
18	1164191339	2769720323	6570216334	15542634372	36671330544
19	487845360	1164191500	2769720849	6570218023	15542639603
20	203760808	487845373	1164191546	2769721010	6570218549
21	84814982	203760809	487845376	1164191559	2769721056
22	35173496	84814982	203760809	487845377	1164191562
23	14530835	35173496	84814982	203760809	487845377
$DS(n)$	171613540653843	376459661897527	825046583681744	1806531250566778	3952141384922689

TABLE 5.16. $DS(n)$ for $46 \leq n \leq 50$ according to the number of connected direct terms $d = m + 1$, where $2 \leq d \leq 26$.

$d \diagdown n$	46	47	48	49	50
2	378232002052118	774056185954327	1583296748191767	3236962232172568	6614661961089048
3	1278365515033104	2679876336571571	5611907339790849	11739852145251492	24535235453168114
4	2084098123128673	4475326479602254	9594966779907938	20540133808336393	43906745522488463
5	2042448364261787	4485095705244385	9829718631564444	21502614372408198	46951849531703518
6	1407602179101315	3149475326523435	7031990909757367	15668563529981774	34843255271767539
7	773694738380576	1755136990705480	3973363951584615	8977075395155599	20242448421527101
8	373729014087492	855379360957198	1954029064304096	4455470993905336	10140647950148663
9	169244336709722	389423469869103	894442022352300	2050808406537735	4694185790941035
10	74462548616777	171899823803530	396146215824697	911383714980428	2093314021306824
11	32359132509046	74880000922228	172970781542159	398880942497005	918335798591009
12	13982605095485	32422030986395	75043211526947	173392205098663	399964009836122
13	6022500819359	13991813705348	32446192869064	75106267106545	173555925826013
14	2587753092221	6023815450370	13995304547710	32455406209237	75130445717662
15	1109500887334	2587935855633	6024307351523	13996619276266	32458897446169
16	474680019773	1109525539678	2588003253544	6024490115887	13997111182819
17	202636702116	474683229043	1109534477863	2588027905888	6024557513798
18	86306175952	202637102413	474684369309	1109537687133	2588036844073
19	36671346455	86306223306	202637241164	474684769606	1109538827399
20	15542641292	36671351686	86306239217	202637288518	474684908357
21	6570218710	15542641818	36671353375	86306244448	202637304429
22	2769721069	6570218756	15542641979	36671353901	86306246137
23	1164191563	2769721072	6570218769	15542642025	36671354062
24	487845377	1164191563	2769721073	6570218772	15542642038
25	203760809	487845377	1164191563	2769721073	6570218773
26	84814982	203760809	487845377	1164191563	2769721073
$DS(n)$	8638765491016575	18867511982595228	41174866181047968	89787245277280689	195646335428736437

TABLE 5.17. $CSP(n)$ for $2 \leq n \leq 16$ according to the number of ordinal terms $t = m + 1$, where $2 \leq t \leq 16$.

$t \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	2	5	16	52	188	690	2638	10272	40782	164114	668544	2750025	11409144	47678552
3		1	3	9	31	108	402	1523	5934	23505	94539	384732	1581819	6559179	27400899
4			1	4	14	52	193	744	2908	11580	46716	190664	785596	3263860	13656666
5				1	5	20	80	315	1261	5085	20730	85260	353525	1476257	6203590
6					1	6	27	116	483	2010	8363	34938	146616	618178	2618268
7						1	7	35	161	707	3059	13132	56253	241003	1033949
8							1	8	44	216	998	4488	19876	87328	382121
9								1	9	54	282	1368	6390	29187	131544
10									1	10	65	360	1830	8872	41780
11										1	11	77	451	2398	12056
12											1	12	90	556	3087
13												1	13	104	676
14													1	14	119
15														1	15
16															1
$CSP(n)$	1	3	9	30	103	375	1400	5380	21073	83950	338878	1383576	5702485	23696081	99163323

TABLE 5.18. $CSP(n)$ for $17 \leq n \leq 23$ according to the number of ordinal terms $t = m + 1$, where $2 \leq t \leq 23$.

$t \diagup n$	17	18	19	20	21	22	23
2	200523288	848079588	3604696476	15389640287	65966258818	283779863972	1224797039140
3	115204380	487115119	2069995539	8835884304	37868209637	162882764373	702919507509
4	57499516	243423630	1035562696	4424662736	18979499816	81701017310	352832260716
5	26214600	111328615	474906920	2034031171	8743566945	37710179635	163133278430
6	11137278	47563411	203876406	876872208	3783262364	16370149059	71023158084
7	4444629	19149403	82698658	357984578	1553201218	6753847884	29430083844
8	1668912	7284896	31806528	138969836	607801416	2661416220	11668455336
9	588033	2615422	11597202	51332706	227004264	1003509306	4436346375
10	193150	882205	3996880	18008635	80836620	361931465	1617718520
11	58509	277420	1294722	5975926	27365635	124602578	564987973
12	16080	80384	390828	1863699	8764276	40796310	188459664
13	3913	21099	108589	541242	2636907	12634440	59786831
14	812	4893	27286	144501	738138	3673635	17933580
15	135	965	6045	34833	189710	992835	5046810
16	16	152	1136	7388	43952	246040	1318752
17	1	17	170	1326	8942	54876	315571
18		1	18	189	1536	10728	67860
19			1	19	209	1767	12768
20				1	20	230	2020
21					1	21	252
22						1	22
23							1
$CSP(n)$	417553252	1767827220	7520966100	32135955585	137849390424	593407692685	2562695780058

TABLE 5.19. $CSP(n)$ for $24 \leq n \leq 26$ according to the number of ordinal terms $t = m + 1$, where $2 \leq t \leq 26$.

$t \diagdown n$	24	25	26
2	5302060266572	23015172766034	100156023119948
3	3042567599614	13205931657918	57463949726325
4	1528218573674	6637012501384	28895896654492
5	707670546150	3077710839138	13416747969945
6	308903412021	1346614757694	5882907093978
7	128499873747	562133029311	2463520268146
8	51224924332	225175178680	991118644552
9	19618314261	86797853991	384256005426
10	7222717450	32226222846	143736900570
11	2554018159	11519524687	51872080623
12	865599511	3958294188	18039819486
13	280245810	1304057287	6033566774
14	86274349	410386326	1934928086
15	25099790	122759184	592609530
16	6845398	34681104	172439368
17	1731688	9177110	47357563
18	400662	2250126	12171438
19	83182	503975	2895562
20	15085	101144	628500
21	2296	17703	122073
22	275	2596	20647
23	23	299	2921
24	1	24	324
25		1	25
26			1
$CSP(n)$	11099806544050	48206136562750	209876865026303

TABLE 5.20. $CSP(n)$ for $27 \leq n \leq 29$ according to the number of ordinal terms $t = m + 1$, where $2 \leq t \leq 29$.

$t \diagdown n$	27	28	29
2	436866300887246	1909648837000588	8364210327184346
3	250630690007218	1095493263961116	4797935382698955
4	126093687133044	551404877911781	2416029938578028
5	58615639707725	256601219306410	1125439210000080
6	25751662141792	112934292889641	496137012745488
7	10814616579494	47551434819762	209399470940681
8	4368029430024	19274601328012	85154606315808
9	1702284021110	7546847139567	33483785796099
10	641031839930	2859032737665	12753899672370
11	233302424014	1048436271333	4708882152865
12	82001661252	371993192448	1684873923096
13	27790483383	127545219269	583700346905
14	9059268148	42177708543	195479817274
15	2831364255	13416081855	63145287960
16	844825584	4090509972	19617698848
17	239370982	1190198492	5840234774
18	63966698	328658112	1658445480
19	15983047	85532813	446676662
20	3692860	20795550	113300480
21	777581	4670634	26825694
22	146322	954943	5861658
23	23943	174271	1164720
24	3272	27618	206328
25	350	3650	31700
26	26	377	4056
27	1	27	405
28		1	28
29			1
$CSP(n)$	915840095739301	4004923697094450	17547807425910789

Table 5.21. $CSP(n)$ for $30 \le n \le 31$ according to the number of ordinal terms $t = m + 1$, where $2 \le t \le 31$.

$t \setminus n$	30	31
2	36703103836745992	161336655247877242
3	21052704408568011	92537103070973520
4	10605462754788320	46633668693043428
5	4944799938990064	21761523742164890
6	2183158457318583	9621295066154820
7	923445625639124	4077909145641268
8	376647885918076	1667818566610048
9	148677656302740	660694836421287
10	56910545193037	254038152328740
11	21141519805186	94900152062826
12	7622109395506	34449348625344
13	2665127656646	12146256058131
14	902642755665	4155418620682
15	295578247831	1377332733915
16	93354480864	441398435536
17	28351120184	136416204846
18	8247464637	40527455328
19	2287402685	11528425427
20	601336704	3124901260
21	148772687	802388517
22	34327986	193754110
23	7303305	43599789
24	1411484	9038016
25	242930	1700275
26	36218	284544
27	4491	41202
28	434	4956
29	29	464
30	1	30
31		1
$CSP(n)$	77027671121229420	338698369075550442

TABLE 5.22. $CSP(n)$ for $32 \leq n \leq 33$ according to the number of ordinal terms $t = m + 1$, where $2 \leq t \leq 33$.

$t \diagdown n$	32	33
2	710341137528953973	3132278654756510024
3	407407869519414738	1796404221628724802
4	205382938000084988	905900743182985776
5	95917741673920015	423386406074242030
6	42462639089725143	187659027664223880
7	18031138077070671	79825035950572836
8	7393143432530994	32806352355054160
9	2938316947918578	13077806078544855
10	1134446054321750	5068312361733590
11	425955681761313	1911927467098816
12	155590341128988	702357688591980
13	55273703474106	251230183742356
14	19082294017269	87448175802204
15	6394020527340	29590140667920
16	2075890299801	9719246439088
17	651580619053	3093109942832
18	197188667064	951535868124
19	57345535007	282156340034
20	15961163780	80371084460
21	4231003686	21900766958
22	1061777970	5680656302
23	250400011	1394057439
24	54986919	321271296
25	11113800	68889775
26	2036632	13584766
27	331668	2426625
28	46683	384832
29	5452	52693
30	495	5980
31	31	527
32	1	32
33		1
$CSP(n)$	1491674669942837919	6579403269510266993

TABLE 5.23. $DSP(n)$ for $2 \le n \le 17$ and $2 \le d \le 17$.

$d \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	1	1	4	12	45	160	613	2354	9297	37118	150369	615092	2540061	10569575	44286415	186666608
3		1	1	4	13	48	175	680	2646	10566	42628	174190	718126	2985818	12499982	52657256
4			1	1	4	13	49	178	695	2713	10873	43987	180270	745092	3105499	13029939
5				1	1	4	13	49	179	698	2728	10940	44294	181650	751307	3133365
6					1	1	4	13	49	179	699	2731	10955	44361	181957	752687
7						1	1	4	13	49	179	699	2732	10958	44376	182024
8							1	1	4	13	49	179	699	2732	10959	44379
9								1	1	4	13	49	179	699	2732	10959
10									1	1	4	13	49	179	699	2732
11										1	1	4	13	49	179	699
12											1	1	4	13	49	179
13												1	1	4	13	49
14													1	1	4	13
15														1	1	4
16															1	1
17																1
$DSP(n)$	1	2	6	18	64	227	856	3280	12885	51342	207544	847886	3497384	14541132	60884173	256480895

TABLE 5.24. $DSP(n)$ for $18 \leq n \leq 24$ according to the number of connected direct terms d, where $2 \leq d \leq 14$.

$d \diagdown n$	18	19	20	21	22	23	24
2	790997237	3367700038	14399128769	61801658911	266177276692	1150041480293	4983225458906
3	223021245	949144261	4056825706	17407092435	74953129228	323772409451	1402671363141
4	55003554	233403698	995091525	4260196175	18307596324	78942369641	341454986527
5	13154855	55562641	235904703	1006273218	4310176483	18530969703	79940691015
6	3139608	13182910	55688862	236471653	1008819448	4321606312	18582272604
7	752994	3140988	13189153	55716953	236598126	1009388183	4324163571
8	182039	753061	3141295	13190533	55723196	236626217	1009514701
9	44380	182042	753076	3141362	13190840	55724576	236632460
10	10959	44380	182043	753079	3141377	13190907	55724883
11	2732	10959	44380	182043	753080	3141380	13190922
12	699	2732	10959	44380	182043	753080	3141381
13	179	699	2732	10959	44380	182043	753080
14	49	179	699	2732	10959	44380	182043
$DSP(n)$	1086310549	4623128656	19759964149	84784735379	365066645854	1576927900803	6831518134251

TABLE 5.25. $DSP(n)$ for $25 \leq n \leq 29$ according to the number of connected direct terms d, where $2 \leq d \leq 15$.

$d \diagdown n$	25	26	27	28	29
2	21650051128260	94290485093409	411582410132646	1800337938433164	7890334515962548
3	6093028529406	26532558602430	115800899755733	506475988605665	2219499028047185
4	1481100103514	6441124261120	28078620644547	122672641179022	537040753000177
5	345917238111	1501049004691	6530329725173	28477639442305	124458064492652
6	80170941222	346950656821	1505687633273	6551153934784	28571143341517
7	18593767365	80222608610	347182892978	1506731601382	6555847529361
8	4324732630	18596326964	80234118104	347234648617	1506964341299
9	1009542792	4324859148	18596896078	80236678108	347246161081
10	236633840	1009549035	4324887239	18597022596	80237247222
11	55724950	236634147	1009550415	4324893482	18597050687
12	13190925	55724965	236634214	1009550722	4324894862
13	3141381	13190926	55724968	236634229	1009550789
14	753080	3141381	13190926	55724969	236634232
15	182043	753080	3141381	13190926	55724969
$DSP(n)$	29674505668536	129216630647787	563949605921815	2466473805675492	10808418041305028

Table 5.26. $DSP(n)$ for $30 \le n \le 33$ according to the number of connected direct terms d, where $2 \le d \le 17$.

$d \diagdown n$	30	31	32	33
2	34643628782312382	152365283679383463	671176028820846781	2960956602723261782
3	9744105111787113	42851636118804922	188748671701174246	832623138772659024
4	2355546232307298	10350101745593312	45552849678933348	200797661057629302
5	545032614086389	2391333057797509	10510416709779197	46271314587716180
6	124878006392422	546919120819290	2399810146123081	10548519903532193
7	28592249360876	124972936119799	547346196364928	2401732005687197
8	6556894286781	28596958006210	124994121760915	547441540401090
9	1507016115913	6557127141880	28598005429585	124998834138925
10	347248721151	1507027628872	6557178920169	28598238308439
11	80237373740	347249290265	1507030188942	6557190433206
12	18597056930	80237401831	347249416783	1507030758056
13	4324895169	18597058310	80237408074	347249444874
14	1009550804	4324895236	18597058617	80237409454
15	236634233	1009550807	4324895251	18597058684
16	55724969	236634233	1009550808	4324895254
17	13190926	55724969	236634233	1009550808
$DSP(n)$	47450298633932617	208667359499177355	919097231317536374	4054308523612570117

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